MOMC IOQM Mock Apple 1

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of a intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- This is Mock Apple 1
- The problems are credited to their respective sources.
- There are a few bonus problems on the last two pages.
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Princeton has an endowment of 5 million dollars and wants to invest it into improving campus life. The university has three options: it can either invest in improving the dorms, campus parties or dining hall food quality. If they invest a million dollars in the dorms, the students will spend an additional $5a$ hours per week studying. If the university invests b million dollars in better food, the students will spend an additional $3b$ hours per week studying. Finally, if the c million dollars are invested in parties, students will be more relaxed and spend $11c - c^2$ more hours per week studying. The university wants to invest its 5 million dollars so that the students get as many additional hours of studying as possible. What is the maximal amount that students get to study?

2. Let ABC be a triangle with $AB = BC$. The altitude from A intersects line BC at D. Suppose $BD = 5$ and $AC^2 = 1188$. Find the sum of all possible values of AB.

3. The expression below has six empty boxes. Each box is to be filled in with a number from 1 to 6 , where all six numbers are used exactly once, and then the expression is evaluated. Let $\frac{m}{n}$ be the maximum possible final result that can be achieved, where m, n are relatively prime positive integers. Find $m + n$.

$$\frac{\frac{\square}{\square} + \frac{\square}{\square}}{\frac{\square}{\square}}$$

4. A robot is at position 0 on a number line. Each second, it randomly moves either one unit in the positive direction or one unit in the negative direction, with probability $\frac{1}{2}$ of doing each. Let the probability that after 4 seconds, the robot has returned to position 0 be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

5. Hexagon $ABCDEF$ has an inscribed circle Ω that is tangent to each of its sides. If $AB = 12$, $\angle FAB = 120°$, and $\angle ABC = 150°$, and if the radius of Ω can be written as $m + \sqrt{n}$ for positive integers m, n, find $m + n$.

SPACE FOR ROUGH WORK

6. How many ordered triples of nonzero integers (a, b, c) satisfy $2abc = a + b + c + 4$?

7. An integer $n > 0$ is such that n when represented in base 2 is written the same way as $2n$ is in base 5. Find the largest possible value of n.

8. Suppose that a, b are positive real numbers with $a > b$ and $ab = 8$. Find the minimum value of $\frac{a^2+b^2}{a-b}$.

9. Let $P(x)$ be a monic polynomial of degree 3. Suppose that $P(x)$ has remainder $R(x)$ when it is divided by $(x-1)(x-4)$ and $2R(x)$ when it is divided by $(x-2)(x-3)$. Given that $P(0) = 5$, find $P(5)$.

10. Suppose S is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. How many different possible values are there for the product of the elements in S ?

SPACE FOR ROUGH WORK

11. Shelly writes down a vector $v = (a, b, c, d)$, where $0 < a < b < c < d$ are integers. Let $\sigma(v)$ denote the set of 24 vectors whose coordinates are a, b, c, and d in some order. For instance, $\sigma(v)$ contains (b, c, d, a). Shelly notes that there are 3 vectors in $\sigma(v)$ whose sum is of the form (s, s, s, s) for some s. What is the smallest possible value of d ?

12. Let S be a square of side length 1 , one of whose vertices is A. Let S^+be the square obtained by rotating S clockwise about A by $30°$. Let S^-be the square obtained by rotating S counterclockwise about A by $30°$. Let the total area that is covered by exactly two of the squares S, S^+, S^- be $\frac{a\sqrt{b}-c}{d}$ where a, c, d are pairwise coprime and b is square-free. Find $a + b + c + d$.

13. In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the integer nearest to 10 times the length of DF.

SPACE FOR ROUGH WORK

14. Let f be a polynomial with integer coefficients such that the greatest common divisor of all its coefficients is 1. For any $n \in \mathbb{N}$, $f(n)$ is a multiple of 85. Find the smallest possible degree of f.

15. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$ and let Q be a 45-degree monic polynomial with roots $\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j}$ where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as $\frac{a}{b}$ for relatively prime integers a, b. Find $a + b$

16. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be a permutation of the numbers $1, 2, 3, 4, 5, 6$. We say a_i is visible if a_i is greater than any number that comes before it; that is, $a_j < a_i$ for all $j < i$. For example, the permutation $2, 4, 1, 3, 6, 5$ has three visible elements: $2, 4, 6$. If there are M permutations that have exactly two visible elements, then find the sum of square of digits of M.

SPACE FOR ROUGH WORK

17. Bob is writing a sequence of letters of the alphabet, each of which can be either uppercase or lowercase, according to the following two rules:

- If he had just written an uppercase letter, he can either write the same letter in lowercase after it, or the next letter of the alphabet in uppercase.
- If he had just written a lowercase letter, he can either write the same letter in uppercase after it, or the preceding letter of the alphabet in lowercase.

For instance, one such sequence is $aAaABCDdcbBC$. Let the number of sequences of 32 letters that he can write that start at (lowercase) a and end at (lowercase) z be N. Find the sum of square of digits of N.

18. A positive integer is called jubilant if the number of 1's in its binary representation is even. For example, $6 = 110_2$ is a jubilant number. What is the sum of square of digits of the 2009^{th} smallest jubilant number?

19. A positive integer x is k-equivocal if there exists two positive integers b, b' such that when x is represented in base b and base b', the two representations have digit sequences of length k that are permutations of each other. The smallest 2-equivocal number is 7 , since 7 is 21 in base 3 and 12 in base 5 . Find the smallest 3-equivocal number.

20. A spider is making a web between $n > 1$ distinct leaves which are equally spaced around a circle. He chooses a leaf to start at, and to make the base layer he travels to each leaf one at a time, making a straight line of silk between each consecutive pair of leaves, such that no two of the lines of silk cross each other and he visits every leaf exactly once. Let f_n be the number of way in which the spider can make the base layer of the web. Find the largest integer n such that $f_n < 2023$.

SPACE FOR ROUGH WORK

21. Circle Ω has radius 13. Circle ω has radius 14 and its center P lies on the boundary of circle Ω. Points A and B lie on Ω such that chord AB has length 24 and is tangent to ω at point T. Find $AT \cdot BT$.

22. The vertices of a regular hexagon are labeled $\cos(\theta), \cos(2\theta), \ldots, \cos(6\theta)$. For every pair of vertices, Benjamin draws a blue line through the vertices if one of these functions can be expressed as a polynomial function of the other (that holds for all real θ), and otherwise Roberta draws a red line through the vertices. In the resulting graph, how many triangles whose vertices lie on the hexagon have at least one red and at least one blue edge?

23. A convex polygon in the Cartesian plane has all of its vertices on integer coordinates. One of the sides of the polygon is AB where $A = (0, 0)$ and $B = (51, 51)$, and the interior angles at A and B are both at most 45 degrees. Assuming no 180 degree angles, what is the maximum number of vertices this polygon can have?

SPACE FOR ROUGH WORK

24. Suppose that a, b, c, d, e, f are real numbers such that

$$\begin{gathered} a+b+c+d+e+f=0 \\ a+2b+3c+4d+2e+2f=0 \\ a+3b+6c+9d+4e+6f=0 \\ a+4b+10c+16d+8e+24f=0 \\ a+5b+15c+25d+16e+120f=42, \end{gathered}$$

Find the sum of digits of $a+6b+21c+36d+32e+720f$.

25. Let a_k be the number of perfect squares m such that $k^3 \leq m < (k+1)^3$. For example, $a_2 = 3$ since three squares m satisfy $2^3 \leq m < 3^3$, namely $9, 16$ and 25. Compute the last two digits of

$$\sum_{k=0}^{99} \lfloor \sqrt{k} \rfloor a_k,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

26. Let p (in base 10) be the third greatest positive integer less than 1000_{10} which is a palindrome in both base 5 and 10. Find the greatest prime divisor of p.

SPACE FOR ROUGH WORK

27. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of a, b, but not both. Let $100m + n$ be the number of pairs of positive integers (x, y), where $x \neq y$, $1 \leq x, y, \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$, where $m, n < 100$ are positive integers. Find $m + n$.

28. A special kind of chess knight is in the origin of an infinite grid. It can make one of twelve different moves: it can move directly up, down, left, or right one unit square, or it can move 1 units in one direction and 3 units in an orthogonal direction. How many different squares can it be on after 2 moves?

29. Katie has a chocolate bar that is a 5×5 grid of square pieces, but she only wants to eat the center piece. To get to it, she performs the following operations:

- Take a gridline on the chocolate bar, and split the bar along the line.
- Remove the piece that doesn't contain the center.
- With the remaining bar, repeat steps 1 and 2.

Let N be the number of ways that Katie can perform this sequence of operations so that eventually she ends up with just the center piece. Find the sum of all distinct primes dividing N.

30. Let $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. How many subsets S of $\{1, 2, \ldots 99\}$ are there such that

$$F_{100} - 1 = \sum_{i \in S} F_i?$$

SPACE FOR ROUGH WORK

Bonus Problems

Note that the answers of the bonus problems may not be in the integer range of 00-99.

2-3 Markers.

1. Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$, and let $S = [f(6)]^5 + [f(10)]^3 + [f(15)]^2$. Compute the remainder when S is divided by 30.

2. In how many distinct ways can you color each of the vertices of a tetrahedron either red, blue, or green such that no face has all three vertices the same color? (Two colorings are considered the same if one coloring can be rotated in three dimensions to obtain the other.)

3. How many ordered quadruples (a, b, c, d) of four distinct numbers chosen from the set $\{1, 2, 3, \ldots, 9\}$ satisfy $b < a, b < c$, and $d < c$?

4. The average of a set of distinct primes is 27. What is the largest prime that can be in this set?

5. Let γ_1 and γ_2 be circles centered at O and P respectively, and externally tangent to each other at point Q. Draw point D on γ_1 and point E on γ_2 such that line DE is tangent to both circles. If the length $OQ = 1$ and the area of the quadrilateral $ODEP$ is 520 , then what is the value of length PQ?

5 Markers.

1. Let $\{x\} = x - \lfloor x \rfloor$. Consider a function f from the set $\{1, 2, \ldots, 2020\}$ to the half-open interval $[0, 1)$. Suppose that for all x, y, there exists a z so that $\{f(x) + f(y)\} = f(z)$. We say that a pair of integers m, n is valid if $1 \leq m, n, \leq 2020$ and there exists a function f satisfying the above so $f(1) = \frac{m}{n}$. Let S be the sum over all valid pairs m, n of $\frac{m}{n}$. Find the sum of digits of S.

2. A circle having radius r_1 centered at point N is tangent to a circle of radius r_2 centered at M. Let l and j be the two common external tangent lines to the two circles. A circle centered at P with radius r_2 is externally tangent to circle N at the point at which l coincides with circle N, and line k is externally tangent to P and N such that points M, N, and P all lie on the same side of k. For what ratio r_1/r_2 are j and k parallel?

3. In a 4×4 grid of sixteen unit squares, exactly 8 are shaded so that each shaded square shares an edge with exactly one other shaded square. How many ways can this be done?

4. Find the number of pairs of positive integers a, b, with $a \leq 125$ and $b \leq 100$, such that $a^b - 1$ is divisible by 125.

5. Let a_k be the number of ordered 10-tuples $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = k.$$

Let $b_k = 0$ if a_k is even and $b_k = 1$ if a_k is odd. Find $\sum_{i=1}^{2012} b_{4i}$.

6. Three different faces of a regular dodecahedron are selected at random and painted. What is the probability that there is at least one pair of painted faces that share an edge?

7. Circle Ω has radius 5. Points A and B lie on Ω such that chord AB has length 6 . A unit circle ω is tangent to chord AB at point T. Given that ω is also internally tangent to Ω, find $AT \cdot BT$.

8. Suppose we have a sequence $a_1, a_2, \ldots$ of positive real numbers so that for each positive integer n, we have that $\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2$. Determine the first value of k so $a_k > 100$.

9. Triangle ABC is so that $AB = 15, BC = 22$, and $AC = 20$. Let D, E, F lie on BC, AC, and AB, respectively, so AD, BE, CF all contain a point K. Let L be the second intersection of the circumcircles of BFK and CEK. Suppose that $\frac{AK}{KD} = \frac{11}{7}$, and $BD = 6$. If $KL^2 = \frac{a}{b}$, where a, b are relatively prime integers, find $a + b$.

10. Let $\mathcal{P}$ be the power set of $\{1, 2, 3, 4\}$ (meaning the elements of $\mathcal{P}$ are the subsets of $\{1, 2, 3, 4\}$). How many subsets S of $\mathcal{P}$ are there such that no two distinct integers $a, b \in \{1, 2, 3, 4\}$ appear together in exactly one element of S ?

11. Cary has six distinct coins in a jar. Occasionally, he takes out three of the coins and adds a dot to each of them. Determine the number of orders in which Cary can choose the coins so that, eventually, for each number $i \in \{0, 1, \ldots, 5\}$, some coin has exactly i dots on it.