MOMC IOQM Mock Apple 4

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. If the radius of ω is $\frac{m}{n}$ where m, n are relatively prime positive integers, then find $m + n$.

2. Find the number of subsets S of $\{1, 2, \ldots 63\}$ the sum of whose elements is 2008.

3. A circle has radius 52 and center O. Points A is on the circle, and point P on $\overline{OA}$ satisfies $OP = 28$. Point Q is constructed such that $QA = QP = 15$, and point B is constructed on the circle so that Q is on $\overline{OB}$. Find QB.

4. Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135^\circ$, then let $BC/AB = \frac{\sqrt{m}}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

5. A particle starts at $(0, 0, 0)$ in three-dimensional space. Each second, it randomly selects one of the eight lattice points a distance of $\sqrt{3}$ from its current location and moves to that point. Let the probability that, after two seconds, the particle is a distance of $2\sqrt{2}$ from its original location be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

6. For how many ordered triples (a, b, c) of positive integers are the equations $abc+9 = ab+bc+ca$ and $a + b + c = 10$ satisfied?

7. Dilhan is running around a track for 12 laps. If halfway through a lap, Dilhan has his phone on him, he has a $\frac{1}{3}$ chance to drop it there. If Dilhan runs past his phone on the ground, he will attempt to pick it up with a $\frac{2}{3}$ chance of success, and won't drop it for the rest of the lap. He starts with his phone at the start of the 5 K, let the chance he still has it when he finished the 5 K be $\frac{a}{b}$ where a, b are relatively prime positive integers. Find $m + n$.

8. How many 4-digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

9. Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1}+\frac{2z-4}{2z^2-5z+1}+\frac{z-2}{z^2-3z+1}=\frac{3}{z}.$$

Over all possible values of z, Let the sum of all the possible values of

$$\left|\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right|$$

be S. Find the integer nearest to $10S$.

10. For polynomials $P(x) = a_n x^n + \cdots + a_0$, let $f(P) = a_n \cdots a_0$ be the product of the coefficients of P. The polynomials P_1, P_2, P_3, Q satisfy $P_1(x) = (x-a)(x-b), P_2(x) = (x-a)(x-c)$, $P_3(x) = (x-b)(x-c), Q(x) = (x-a)(x-b)(x-c)$ for some complex numbers a, b, c. Given $f(Q) = 8, f(P_1) + f(P_2) + f(P_3) = 10$, and $abc > 0$, find the value of $f(P_1)\, f(P_2)\, f(P_3)$.

SPACE FOR ROUGH WORK

11. Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many minutes does it take Trodgor to burn down the entire village?

12. Let $P(x)$ be a polynomial with degree 4 and leading coefficient 1 such that

$$P(0) = 3, P(1) = 2, P(2) = 1, P(3) = 0$$

Determine the value of $P(4)$.

13. Alan is assigning values to lattice points on the 3D coordinate plane. First, Alan computes the roots of the cubic $20x^3 - 22x^2 + 2x + 1$ and finds that they are α, β, and γ. He finds out that each of these roots satisfy $|\alpha|, |\beta|, |\gamma| \leq 1$ On each point (x, y, z) where x, y, and z are all nonnegative integers, Alan writes down $\alpha^x \beta^y \gamma^z$. What is the value of the sum of all numbers he writes down?

14. For a positive integer n, let $\theta(n)$ denote the number of integers $0 \leq x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 100.

15. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

SPACE FOR ROUGH WORK

16. We say that a set S of 3 unit squares is commutable if $S = \{s_1, s_2, s_3\}$ for some s_1, s_2, s_3 where s_2 shares a side with each of s_1, s_3. How many ways are there to partition a 3×3 grid of unit squares into 3 pairwise disjoint commutable sets?

17. Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Let the x-coordinate of the point X on the x-axis maximizing $\angle AXB$ be $a\sqrt{b} - c$ where a, b, c are positive integers such that b is square-free. Find $a + b + c$.

18. Let ABC be an acute triangle with $\angle ABC = 60^\circ$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE = 45$, what is the integer nearest to $\sqrt{AB \cdot BC}$?

19. Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

20. A shipping company charges $.30l + .40w + .50h$ dollars to process a right rectangular prism-shaped box with dimensions l, w, h in inches. The customers themselves are allowed to label the three dimensions of their box with l, w, h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of $\$8.10$, and two different ways to label B to get a fee of $\$8.70$. None of the faces of B are squares. Find the sum of the distinct prime factors of the volume of B.

SPACE FOR ROUGH WORK

21. A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. Find the remainder when $M + N$ is divided by 100. (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$)

22. Find the largest prime factor of the smallest positive integer N such that each of the 101 intervals

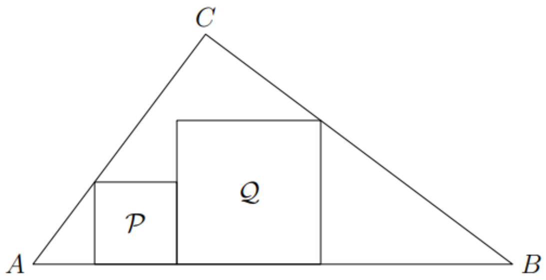
$$\left[N^2, (N+1)^2\right), \left[(N+1)^2, (N+2)^2\right), \cdots, \left[(N+100)^2, (N+101)^2\right)$$

contains at least one multiple of 1001.

23. Let F_n denote the nth Fibonacci number, with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exists a unique two digit prime p such that for all $n, p \mid F_{n+100} + F_n$. Find p.

SPACE FOR ROUGH WORK

24. Let ABC be a triangle with $AB = 5, BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Let the minimum value of the sum of the areas of the two squares be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $|m - n|$.



25. Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, let the maximum possible area of ABC be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

26. A triangle $\triangle ABC$ satisfies $AB = 13, BC = 14$, and $AC = 15$. Inside $\triangle ABC$ are three points X, Y, and Z such that:

- Y is the centroid of $\triangle ABX$
- Z is the centroid of $\triangle BCY$
- X is the centroid of $\triangle CAZ$

Let the area of $\triangle XYZ$ be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

27. Daniel, Ethan, and Zack are playing a multi-round game of Tetris. Whoever wins 11 rounds first is crowned the champion. However Zack is trying to pull off a "reverse-sweep", where (at-least) one of the other two players first hits 10 wins while Zack is still at 0, but Zack still ends up being the first to reach 11. Find the last two digits of the possible sequences of round wins that can lead to Zack pulling off a reverse sweep.

28. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

29. There are 9 points arranged in a 3×3 square grid. Let two points be adjacent if the distance between them is half the side length of the grid. (There should be 12 pairs of adjacent points). Suppose that we wanted to connect 8 pairs of adjacent points, such that all points are connected to each other. Suppose that there are M ways in which this is possible. Find the sum of square of digits of M.

30. Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny (worth 1 cent), which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel (worth 5 cents), which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

SPACE FOR ROUGH WORK