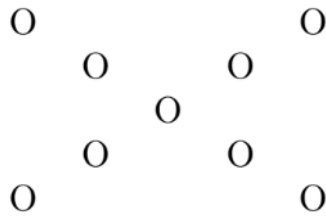


# MOMC IOQM Mock Blueberry 3

Instructions:

- All answers are in the integer range of 00 – 99. Although there is a non-zero chance of an intentional bonus.
- Problems 1 – 10 are 2 Markers, 11 – 20 are 3 Markers and 21 – 30 are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- After finishing the mock and checking your answers, **be sure to fill in [this form](#) about the mock.** I will really appreciate it!
- Good luck!

1. Given that there exists a unique 3 digit number  $N = \overline{ABC}$ , whose digits  $(A, B, C)$  are all nonzero, with the property that the product  $P = \overline{ABC} \times \overline{AB} \times \overline{A}$  is divisible by 1000, find  $\overline{BC}$ .
2. How many multiples of 12 divide  $12!$  and have exactly 12 divisors?
3. Triangle  $ABC$  has a right angle at  $A$ ,  $AB = 20$ , and  $AC = 21$ . Circles  $\omega_A, \omega_B$ , and  $\omega_C$  are centered at  $A, B$ , and  $C$  respectively and pass through the midpoint  $M$  of  $\overline{BC}$ .  $\omega_A$  and  $\omega_B$  intersect at  $X \neq M$ , and  $\omega_A$  and  $\omega_C$  intersect at  $Y \neq M$ . Find  $XY$ .
4. The incircle of  $\triangle ABC$  is tangent to sides  $\overline{BC}, \overline{AC}$ , and  $\overline{AB}$  at  $D, E$ , and  $F$ , respectively. Point  $G$  is the intersection of lines  $AC$  and  $DF$  as shown. The sides of  $\triangle ABC$  have lengths  $AB = 73, BC = 123$ , and  $AC = 120$ . Find the length  $EG$ .
5. We have a 9 by 9 chessboard with 9 kings (which can move to any of 8 adjacent squares) in the bottom row. What is the minimum number of moves, if two pieces cannot occupy the same square at the same time, to move all the kings into an  $X$  shape (a  $5 \times 5$  region where there are 5 kings along each diagonal of the  $X$ , as shown below)?




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**SPACE FOR ROUGH WORK**

6. Dilhan has objects of 3 types,  $A$ ,  $B$ , and  $C$ , and 6 functions

$$f_{A,B}, f_{A,C}, f_{B,A}, f_{B,C}, f_{C,A}, f_{C,B}$$

where  $f_{X,Y}$  takes in an object of type  $X$  and outputs an object of type  $Y$ . Dilhan wants to compose his 6 functions, without repeats, such that the resulting expression is well-typed, meaning an object can be taken in by the first function, and the resulting output can then be taken in by the second function, and so on. In how many orders can he compose his 6 functions, satisfying this constraint?

7. Given a trapezoid with bases  $AB$  and  $CD$ , there exists a point  $E$  on  $CD$  such that drawing the segments  $AE$  and  $BE$  partitions the trapezoid into 3 similar isosceles triangles, each with long side twice the short side. What is the integer nearest to the sum of all possible values of  $\frac{CD}{AB}$ ?

8. Consider all ordered pairs  $(m, n)$  of positive integers satisfying  $59m - 68n = mn$ . Find the largest prime factor of the sum of all the possible values of  $n$  in these ordered pairs.

9. Pam lists the four smallest positive prime numbers in increasing order. When she divides the positive integer  $N$  by the first prime, the remainder is 1. When she divides  $N$  by the second prime, the remainder is 2. When she divides  $N$  by the third prime, the remainder is 3. When she divides  $N$  by the fourth prime, the remainder is 4. Find the least possible value for  $N$ .

10. The side lengths of a scalene triangle are roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the integer nearest to the area of the triangle.

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**SPACE FOR ROUGH WORK**

**11.** What is the largest prime factor of the  $101^{\text{st}}$  smallest integer which can be represented in the form  $3^a + 3^b + 3^c$ , where  $a, b$ , and  $c$  are integers?

**12.** Let  $a$  and  $b$  be complex numbers such that  $(a + 1)(b + 1) = 2$  and  $(a^2 + 1)(b^2 + 1) = 32$ . Let the sum of all possible values of  $(a^4 + 1)(b^4 + 1)$  be  $S$ . Find the integer nearest to  $\sqrt{S}$ .

**13.** Points  $A, B$ , and  $C$  lie on a line, in that order, with  $AB = 8$  and  $BC = 2$ .  $B$  is rotated  $20^\circ$  counterclockwise about  $A$  to a point  $B'$ , tracing out an arc  $R_1$ .  $C$  is then rotated  $20^\circ$  clockwise about  $A$  to a point  $C'$ , tracing out an arc  $R_2$ . Let the area of the region bounded by arc  $R_1$ , segment  $B'C$ , arc  $R_2$ , and segment  $C'B$  be  $\mathcal{A}$ . Find the integer nearest to  $10\mathcal{A}$ .

**14.** Consider trapezoid  $[ABCD]$  which has  $AB \parallel CD$  with  $AB = 5$  and  $CD = 9$ . Moreover,  $\angle C = 15^\circ$  and  $\angle D = 75^\circ$ . Let  $M_1$  be the midpoint of  $AB$  and  $M_2$  be the midpoint of  $CD$ . What is the distance  $M_1M_2$ ?

**15.** Let  $p_1, p_2, p_3, p_4, p_5, p_6$  be distinct primes greater than 5. Find the minimum possible value of

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6 \min(p_1, p_2, p_3, p_4, p_5, p_6)$$

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**SPACE FOR ROUGH WORK**

**16.** Suppose there are 160 pigeons and  $n$  holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the  $i$ th pigeon flies to the  $(i^2 \bmod n)$ th hole, where  $k \bmod n$  is the remainder when  $k$  is divided by  $n$ . What is the sum of square of digits of the minimum  $n$  such that there is at most one pigeon per hole?

**17.** Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. Suppose there are  $N$  selections of 5 balls that he could have drawn from the box. Find the sum of digits of  $N$ .

**18.** Bill Gates and Jeff Bezos are playing a game. Each turn, a coin is flipped, and if Bill and Jeff have  $m, n > 0$  dollars, respectively, the winner of the coin toss will take  $\min(m, n)$  from the loser. Given that Bill starts with 20 dollars and Jeff starts with 21 dollars, let the probability that Bill ends up with all of the money be  $\frac{a}{b}$  where  $a, b$  are relatively prime positive integers. Find  $a + b$ .

**19.** Let  $ABC$  be a triangle with circumcenter  $O$ . Additionally,  $\angle BAC = 20^\circ$  and  $\angle BCA = 70^\circ$ . Let  $D, E$  be points on side  $AC$  such that  $BO$  bisects  $\angle ABD$  and  $BE$  bisects  $\angle CBD$ . If  $P$  and  $Q$  are points on line  $BC$  such that  $DP$  and  $EQ$  are perpendicular to  $AC$ , what is  $\angle PAQ$ ?

**20.** Let

$$\sum_{i=0}^{\infty} \frac{7^i}{(7^i + 1)(7^i + 7)} = \frac{m}{n}$$

where  $m, n$  are relatively prime positive integers. Find  $m + n$ .

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**SPACE FOR ROUGH WORK**

**21.** For each positive integer  $n$ , let  $\sigma(n)$  denote the sum of the positive integer divisors of  $n$ . Suppose there exist  $M$  positive integers  $n \leq 2021$  that satisfy

$$\sigma(3n) \geq \sigma(n) + \sigma(2n)$$

Find the remainder when  $M$  is divided by 100.

**22.** Let  $f(x) = \frac{x^2}{8}$ . Starting at the point  $(7, 3)$ , let the length of the shortest path that touches the graph of  $f$ , and then the  $x$ -axis be  $a\sqrt{b} - c$  where  $a, b, c$  are positive integers such that  $b$  is square-free. Find  $a + b + c$ .

**23.** Let the remainder when

$$\left\lfloor \frac{149^{151} + 151^{149}}{22499} \right\rfloor$$

is divided by  $10^4$  be  $100p + q$  where  $p, q < 100$  are positive integers. Find  $p + q$ .

**24.** A  $2\sqrt{5}$  by  $4\sqrt{5}$  rectangle is rotated by an angle  $\theta$  about one of its diagonals. If the total volume swept out by the rotating rectangle is  $62\pi$ , find the sum of square of digits of the measure of  $\theta$  in degrees.

**25.** Find the last two digits of the number of permutations of the string 0123456 that are there such that no contiguous substrings of lengths  $1 < \ell < 7$  have a sum of digits divisible by 7?

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**SPACE FOR ROUGH WORK**

**26.** As a gift, Dilhan was given the number  $n = 1^1 \cdot 2^2 \cdot \dots \cdot 2021^{2021}$ , and each day, he has been dividing  $n$  by  $2021!$  exactly once. One day, when he did this, he discovered that, for the first time,  $n$  was no longer an integer, but instead a reduced fraction of the form  $\frac{a}{b}$ . What are the last two digits of the sum of all distinct prime factors of  $b$ ?

**27.** In convex quadrilateral  $ABCD$ ,  $\angle ADC = 90^\circ + \angle BAC$ . Given that  $AB = BC = 17$ , and  $CD = 16$ , what are the last two digits of the largest integer less than the maximum possible area of the quadrilateral?

**28.** Let  $P(x)$ ,  $Q(x)$ , and  $R(x)$  be three monic quadratic polynomials with only real roots, satisfying

$$P(Q(x)) = (x - 1)(x - 3)(x - 5)(x - 7)$$

$$Q(R(x)) = (x - 2)(x - 4)(x - 6)(x - 8)$$

for all real numbers  $x$ . What is  $P(0) + Q(0) - R(0)$ ?

**29.** Let  $\Gamma_1, \Gamma_2, \Gamma_3$  be three circles with radii 3, 4, 9, respectively, such that  $\Gamma_1$  and  $\Gamma_2$  are externally tangent at  $C$ , and  $\Gamma_3$  is internally tangent to  $\Gamma_1$  and  $\Gamma_2$  at  $A$  and  $B$ , respectively. Suppose the tangents to  $\Gamma_3$  at  $A$  and  $B$  intersect at  $X$ . The line through  $X$  and  $C$  intersect  $\Gamma_3$  at two points,  $P$  and  $Q$ . Let  $PQ = \frac{a\sqrt{b}}{c}$  where  $a, b, c$  are positive integers such that  $a+b+c$  is minimal. Find  $a+b+c$ .

**30.** Find the sum of digits of the number of functions  $f : \{1, 2, 3, \dots, 7\} \rightarrow \{1, 2, 3, \dots, 7\}$  that are there such that the set  $\mathcal{F} = \{f(i) : i \in \{1, \dots, 7\}\}$  has cardinality four, while the set  $\mathcal{G} = \{f(f(f(i))) : i \in \{1, \dots, 7\}\}$  consists of a single element.

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**SPACE FOR ROUGH WORK**