MOMC IOQM Mock Blueberry 4

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Find the number of ways to color a 2×2 grid of squares with 4 colors such that no two (nondiagonally) adjacent squares have the same color. Each square should be colored entirely with one color. Colorings that are rotations or reflections of each other should be considered different.

2. Niharika is bored in class, so she thinks of a positive integer. Every second after that, she subtracts from her current number its smallest prime divisor, possibly itself. After 2022 seconds, she realizes that her number is prime. Let the sum of all possible values of her initial number be S. Find the sum of digits of S.

3. Niharika is once again very bored in class. On a whim, she chooses three primes p, q, r independently and uniformly at random from the set of primes of at most 30. She then calculates the roots of $px^2 + qx + r$. Let the probability that at least one of her roots is an integer be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find the integer nearest to $m + \sqrt{n}$.

4. Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and F be the midpoint of CD. Points P and Q are on segments EF and CF, respectively, such that A, P, and Q are collinear. Given that $EP = 5, PF = 3$, and $QF = 12$, find CQ.

5. If

$$f(x) = \frac{2^{19}x + 2^{20}}{x^2 + 2^{20}x + 2^{20}},$$

find the largest positive integer less than $f(1) + f(2) + f(4) + f(8) + \cdots + f(2^{20})$.

SPACE FOR ROUGH WORK

6. Vikram and Betal are playing in an eight-player single-elimination rock-paper-scissors tournament. In the first round, all players are paired up randomly to play a match. Each round after that, the winners of the previous round are paired up randomly. After three rounds, the last remaining player is considered the champion. Ties are broken with a coin flip. Given that Vikram always plays rock, Betal always plays paper, and everyone else always plays scissors, let the probability that Vikram is crowned champion be $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $p+q$. Note that rock beats scissors, scissors beats paper, and paper beats rock.

7. What is the integer nearest to 10 times the smallest r such that three disks of radius r can completely cover up a unit disk?

8. Let n be the answer to this problem. In acute triangle ABC, point D is located on side BC so that $\angle BAD = \angle DAC$ and point E is located on AC so that $BE \perp AC$. Segments BE and AD intersect at X such that $\angle BXD = n^\circ$. Given that $\angle XBA = 16^\circ$, find the measure of $\angle BCA$.

9. Right triangle ABC has $AB = 5, BC = 12$, and $CA = 13$. Point D lies on the angle bisector of $\angle BAC$ such that CD is parallel to AB. Find the integer nearest to BD.

10. How many integers n greater than 2 are there such that the degree measure of each interior angle of a regular n-gon is an even integer?

SPACE FOR ROUGH WORK

11. Malay and Ravi stand atop two different towers in the Arctic. Both towers are a positive integer number of meters tall and are a positive (not necessarily integer) distance away from each other. One night, the sea between them has frozen completely into reflective ice. Malay shines his flashlight directly at the top of Ravi's tower, and Ravi shines his flashlight at the top of Malay's tower by first reflecting it off the ice. The light from Malay's tower travels 16 meters to get to Ravi's tower, while the light from Ravi's tower travels 26 meters to get to Malay's tower. Assuming that the lights are both shone from exactly the top of their respective towers, what is the sum of all the possibilities for the height of Malay's tower?

12. You start with a single piece of chalk of length 1. Every second, you choose a piece of chalk that you have uniformly at random and break it in half. You continue this until you have 8 pieces of chalk. Let the probability that they all have length $\frac{1}{8}$ be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.

13. In triangle ABC, $\angle ABC = 3\angle ACB$. If $AB = 4$ and $AC = 5$, find the integer nearest to $10BC$.

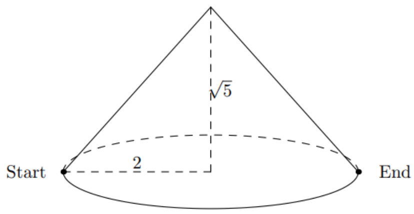
SPACE FOR ROUGH WORK

14. Let $f(x) = x^3 + 3x - 1$ have roots a, b, c. Given that

$$\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3}$$

can be written as $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$, find $|m - n|$.

15. Aarav the ant sits on the circumference of the circular base of a party hat (a cone without a circular base for the ant to walk on) of radius 2 and height $\sqrt{5}$. If the ant wants to reach a point diametrically opposite of its current location on the hat, what is the square of the minimum possible distance the ant needs to travel?



16. Let

$$a = \underbrace{19191919\ldots1919}_{19 \text{ is repeated } 3838 \text{ times}} .$$

What is the remainder when a is divided by 13 ?

SPACE FOR ROUGH WORK

17. Aadi draws a quadrilateral with side lengths $7, 15, 20$, and 24 in some order such that the quadrilateral has two opposite right angles. Let the area of the quadrilateral be $\mathcal{A}$. Find the integer nearest to $\mathcal{A}/10$.

18. Let $\{a_i\}$ for $1 \leq i \leq 10$ be a finite sequence of 10 integers such that for all odd $i, a_i = 1$ or -1 , and for all even $i, a_i = 1, -1$, or 0. Suppose there are $100m + n$ sequences $\{a_i\}$ such that $a_1 + a_2 + a_3 + \cdots + a_{10} = 0$, where $m, n < 100$ are positive integers. Find $m + n$.

19. Positive integers $a \leq b \leq c$ have the property that each of $a + b, b + c$, and $c + a$ are prime. If $a + b + c$ has exactly 4 positive divisors, let $\mathcal{P}$ be the third smallest possible value of the product $c(c + b)(c + b + a)$. Find the sum of digits of $\mathcal{P}$.

20. S is a collection of integers n with $1 \leq n \leq 50$ so that each integer in S is composite and relatively prime to every other integer in S. What is the largest possible number of integers in S ?

SPACE FOR ROUGH WORK

21. A regular octagon is inscribed in a circle of radius 2. Motu and Patlu play a game in which they take turns claiming vertices of the octagon, with Motu going first. A player wins as soon as they have selected three points that form a right angle. If all points are selected without either player winning, the game ends in a draw. Given that both players play optimally, let the sum of all possible areas of the convex polygon formed by Motu's points at the end of the game be $a + \sqrt{b}$ where a, b are positive integers. Find $a + b$.

22. Agamjeet is camping by himself in the forest. He has six wooden sticks of length $1, 2, 3, 4, 5, 6$ inches. Somehow, he has managed to arrange these sticks, such that they form the sides of an equiangular hexagon. Find the integer nearest to the sum of all possible values of the area of this hexagon.

23. Imran and Krishiv play the following "point guessing game". First, Imran marks an equilateral triangle ABC and a point D on segment BC satisfying $BD = 3$ and $CD = 5$. Then, Imran chooses a point P on line AD and challenges Krishiv to mark a point $Q \neq P$ on line AD such that $\frac{BQ}{QC} = \frac{BP}{PC}$. Imran wins if and only if Krishiv is unable to choose such a point. If Imran wins, let S be the sum of all the possible values of $\frac{BP}{PC}$ for the P he choses. Find the largest integer less than $10S$.

SPACE FOR ROUGH WORK

24. Consider parallelogram $ABCD$ with $AB > BC$. Point E on $\overline{AB}$ and point F on $\overline{CD}$ are marked such that there exists a circle ω_1 passing through A, D, E, F and a circle ω_2 passing through B, C, E, F. If ω_1, ω_2 partition $\overline{BD}$ into segments $\overline{BX}, \overline{XY}, \overline{YD}$ in that order, with lengths $200, 9, 80$, respectively, compute BC.

25. The taxicab distance between points (x_1, y_1) and (x_2, y_2) is $|x_2 - x_1| + |y_2 - y_1|$. A regular octagon is positioned in the xy plane so that one of its sides has endpoints $(0, 0)$ and $(1, 0)$. Let S be the set of all points inside the octagon whose taxicab distance from some octagon vertex is at most $\frac{2}{3}$. The area of S can be written as $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $m+n$.

26. A triple of positive integers (a, b, c) is tasty if $\operatorname{lcm}(a, b, c) \mid a + b + c - 1$ and $a < b < c$. Find the sum of $a + b + c$ across all tasty triples.

SPACE FOR ROUGH WORK

27. Debayu thinks of four positive integers $a \le b \le c \le d$ satisfying $\{ab + cd, ac + bd, ad + bc\} = \{40, 70, 100\}$. Find the largest possible value of $abc + d$.

28. Compute the number of ordered pairs of positive integers (a, b) satisfying the equation

$$\gcd(a, b) \cdot a + b^2 = 10000.$$

29. How many ways are there to place 31 knights in the cells of an 8×8 unit grid so that no two attack one another? (A knight attacks another knight if the distance between the centers of their cells is exactly $\sqrt{5}$.)

30. Find the largest prime factor of the number of 10-digit numbers $\overline{a_1 a_2 \cdots a_{10}}$ which are multiples of 11 such that the digits are non-increasing from left to right, i.e. $a_i \ge a_{i+1}$ for each $1 \le i \le 9$.

SPACE FOR ROUGH WORK