MOMC IOQM Mock Cake 1

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- After finishing the mock and checking your answers, **be sure to fill in** this **form about the mock.** I will really appreciate it!
- Good luck and enjoy the CAKE!!

1. Consider the system of equations

$$20\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{y}{2} \right\rfloor = 103$$
$$21\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{y}{2} \right\rceil = 109$$

where x and y are positive integers. Compute the least possible value of $x + y$.

2. Starting at $(0, 0)$, a frog moves in the coordinate plane via a sequence of hops. Each hop is either 1 unit in the x-direction or 1 unit in the y-direction. Compute the minimum number of hops needed for the frog to land on the line $15x + 35y = 2020$.

3. Compute the number of positive integers $b \geq 5$ for which the base-ten number 2020 ends in the digit 4 when it is expressed in base b.

4. A microwave accepts three digits a, b, and c, not all zero, as input. The microwave will then run for $60a + 10b + c$ seconds. Find the sum of digits of the number of positive integers k such that the microwave can run for exactly k seconds.

5. Let $TRIANGLE$ be an equilateral octagon with side length 10, and let α be the acute angle whose tangent is $\frac{3}{4}$. Given that the measures of the interior angles of $TRIANGLE$ alternate between $180^\circ - \alpha$ and $90^\circ + \alpha$, let the area of $TRIANGLE$ be $\mathcal{A}$. Find the integer nearest to $\mathcal{A}/10$.

6. Suppose that $\triangle ABC$ is a right triangle with hypotenuse 20 and legs of length at least 10. Compute the number of possible integer values for $[ABC]$.

7. Compute the sum of square of digits of the greatest integer that is the least common multiple of two distinct sets of four nonzero digits.

8. Compute the number of elements n of the set $\{1, 2, 3, \ldots, 100\}$ for which the product $(n^2 - n + 3)(n^2 + n + 3)$ is a multiple of 5.

9. Arnav draws an n-gon and notices that the degree measure of each interior angle is an integer multiple of 54° and that not all interior angles have the same measure. Compute the least possible value of n.

10. The sum $\sum_{n=3}^{10} \frac{2}{n(n^2-1)}$ is equal to $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $q - p$.

SPACE FOR ROUGH WORK

11. A circle passes through both trisection points of side $\overline{AB}$ of square $ABCD$ and intersects $\overline{BC}$ at points P and Q. Let the greatest possible value of $\tan \angle PAQ$ be $\frac{a}{b}$ where a, b are relatively prime positive integers. Find $a + b$.

12. Let N be the least integer $n > 2020$ such that $(n + 2020)^{n-2020}$ divides n^n. Find $N - 2020$.

13. In $\triangle ABC, AB = 6, BC = 7$, and $CA = 8$. The circle that passes through each of the vertices of $\triangle ABC$ is centered at point O. The circle that passes through each of the vertices of $\triangle AOB$ intersects $\overline{BC}$ at point D, different from B. Let $BD = \frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

14. A set $\mathcal{U}$ has n elements. Let $f(n)$ denote the sum of $|\mathcal{S}|$ over all subsets $\mathcal{S} \subseteq \mathcal{U}$. (Note: For a finite set $\mathcal{A}$, the notation $|\mathcal{A}|$ represents the number of elements of $\mathcal{A}$.) Find the least positive integer n such that $f(n) > 2023$.

15. In hexagon $SQUARE$, all sides have length $10, \angle Q$ and $\angle R$ are right angles, and $\angle S \cong \angle U \cong \angle A \cong \angle E$. Given that the area of $SQUARE$ in simplest form is uniquely expressible as $p + q\sqrt{r}$, where p and q are integers and r is a prime, find the sum of digits of $p + q + r$.

SPACE FOR ROUGH WORK

16. Let $100a + b$ be the remainder when 111^{2021} is divided by 10000, where $a, b < 100$ are positive integers. Find $a + b$.

17. A base-b palindrome is a base-b integer whose digits read the same forwards and backwards and whose leftmost digit is nonzero. Let the greatest base-10 palindrome less than 500 that is also a palindrome when converted to base-2 be $\overline{mnp}$. Find $m^2 + n + p^2$.

18. A circle of integer radius r has a chord PQ of length 8. There is a point X on chord PQ such that $\overline{PX} = 2$ and $\overline{XQ} = 6$. Call a chord AB computational if it contains X and both $\overline{AX}$ and $\overline{XB}$ are integers. What is the minimal possible integer r such that there exist 6 computational chords for X ?

19. On planet MOMC, two individuals may play a game where they write the number 324000 on a whiteboard and take turns dividing the number by prime powers - numbers of the form p^k for some prime p and positive integer k. Divisions are only legal if the resulting number is an integer. The last player to make a move wins. Determine what number the first player should select to divide 324000 by in order to ensure a win.

20. Kartik has 50 coins that total to a value of 82 cents. Tanmay randomly steals one coin and finds out that he took a quarter. As to not enrage Kartik, he quickly puts the coin back into the collection. However, he is both bored and curious and decide to randomly take another coin. Let the probability that this next coin is a penny be p. Find the integer nearest to $100p$. (Every coin is either a penny, nickel, dime or quarter).

SPACE FOR ROUGH WORK

21. The latus rectum of a parabola is the line segment parallel to the directrix, with endpoints on the parabola, that passes through the focus. Define the latus nextum of a parabola to be the line segment parallel to the directrix, with endpoints on the parabola, that is twice as long as the latus rectum. Let the greatest possible distance between a point on the latus rectum and a point on the latus nextum of the parabola whose equation is $y = x^2 + 20x + 20$ be $\frac{m\sqrt{n}}{p}$ where m, n, p are positive integers such that $m + n + p$ is minimal. Find $m + n + p$.

22. Let $ABCDE$ be a convex pentagon inscribed in a circle. The arcs AB, BC, CD, DE, and EA have measures $32°, 64°, 84°, 84°$, and $96°$, respectively. Lines BC and AD meet at point X, lines AB and DE meet at point Y, and point M is the midpoint of $\overline{DY}$. The degree measure of the acute angle formed by the intersection of lines XM and AB is $T°$. Compute T.

23. Circle Ω_1 with radius 11 and circle Ω_2 with radius 5 are externally tangent. Circle Γ is internally tangent to both Ω_1 and Ω_2, and the centers of all three circles are collinear. Line ℓ is tangent to Ω_1 and Ω_2 at distinct points D and E, respectively. Point F lies on Γ so that $FD < FE$ and $\mathrm{m}\angle DFE = 90°$. Find the integer nearest to $100 \sin \angle DEF$.

SPACE FOR ROUGH WORK

24. A finite set of distinct, nonnegative integers $\{a_1, \ldots, a_k\}$ is called admissible if the integer function $f(n) = (n + a_1) \cdots (n + a_k)$ has no common divisor over all terms; that is, $\gcd(f(1), f(2), \ldots, f(n)) = 1$ for any integer n. How many admissible sets only have members of value less than 10? $\{4\}$ and $\{0, 2, 6\}$ are such sets, but $\{4, 9\}$ and $\{1, 3, 5\}$ are not.

25. For a positive integer n, let $s(n)$ denote the sum of the digits of n. Compute the sum of digits of the sum of all positive integers n for which $n = 37s(n)$.

26. Compute the number of ordered pairs (a, b) of integers such that $|a| \leq 2020, |b| \leq 2020$, and

$$\sqrt[3]{\frac{a+b}{2}} = \left(\frac{a}{2}\right)^3 + \frac{b}{2}.$$

SPACE FOR ROUGH WORK

27. Let $a_1, a_2, a_3, a_4, \ldots$ be the increasing sequence $4, 5, 44, 45, \ldots$ that consists of all positive integers each of whose digits is either 4 or 5. Find the remainder when a_{100} is divided by 97.

28. Find the sum of digits of the number of subsets S of $\{1, 2, \ldots, 15\}$ such that S has at least two elements and no two elements of S are relatively prime.

29. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be standard, fair, six-sided dice. Let $\mathcal{D}_3$ be a fair six-sided die with the numbers $1, 2, 2, 3, 3, 4$ on its faces. Let $\mathcal{D}_4$ be a fair six-sided die such that for all integers k between 2 and 12 inclusive, the probability that the sum of a roll of $\mathcal{D}_3$ and $\mathcal{D}_4$ equals k is equal to the probability that the sum of a roll of $\mathcal{D}_1$ and $\mathcal{D}_2$ equals k. Two of the four dice are chosen at random. The two dice are rolled independently of one another. Let the probability that the positive difference of the numbers rolled is 2 be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $|m - n|$.

30. Compute the sum of square of digits of the least positive integer n such that each of $20n$ and $21n$ has exactly 60 positive divisors.

SPACE FOR ROUGH WORK