MOMC IOQM Mock Cake 2

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- After finishing the mock and checking your answers, **be sure to fill in** this **form about the mock.** I will really appreciate it!
- Good luck and enjoy the CAKE!!!

1. There are 100 people in a room with ages $1, 2, \ldots, 100$. A pair of people is called cute if each of them is at least seven years older than half the age of the other person in the pair. At most how many pairwise disjoint cute pairs can be formed in this room?

2. The pairwise products ab, bc, cd, and da of positive integers a, b, c, and d are $64, 88, 120$, and 165 in some order. Find $a + b + c + d$.

3. Given two distinct points A, B and line ℓ that is not perpendicular to AB, what is the maximum possible number of points P on ℓ such that ABP is an isosceles triangle?

4. In the Year 0 of Cambridge there is one squirrel and one rabbit. Both animals multiply in numbers quickly. In particular, if there are m squirrels and n rabbits in Year k, then there will be $2m + 2019$ squirrels and $4n - 2$ rabbits in Year $k + 1$. What is the first year in which there will be strictly more rabbits than squirrels?

5. Let x and y be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

SPACE FOR ROUGH WORK

6. Call a positive integer n Indian if n does not divide $(n-2)!$. Determine the number of Indian numbers between 2 and 100 inclusive.

7. What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is palindromic if the sequence of decimal digits are the same when read backwards.)

8. A non-negative integer is called falling if its digits are strictly decreasing from left to right. 9876543210 is the largest falling number, 0 is the smallest. If all of the falling numbers were written in a list from smallest to largest, let the 1000^{th} number on that list be $\overline{a_1a_2\cdots a_{2n}}$ for some positive integer n. Let $S = a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n}$. Find the sum of square of digits of S.

9. For a positive integer n, let $p(n)$ be the product of the digits of n and $s(n)$ be the sum of the digits of n. Compute the number of values of n less than 100 such that $n = p(n) + s(n)$.

10. Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths $2, 7, 5, 12$, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC. Find $M^2 - m^2$.

SPACE FOR ROUGH WORK

11. A regular hexagon $PROFIT$ has area 1. Every minute, greedy George places the largest possible equilateral triangle that does not overlap with other already-placed triangles in the hexagon, with ties broken arbitrarily. How many triangles would George need to cover at least 90% of the hexagon's area?

12. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the integer nearest to the greatest possible value of a_4b_4.

13. Compute the number of ordered triples of integers (x, y, z) such that $xy - |z| = 20$ and $|x + y| - z = 15$

14. Define a positive integer N to be lucky if there exists a positive integer a such that a^7 has exactly N positive divisors. Compute the number of positive integers less than 100 that are lucky.

15. A deck of 3 red cards, 3 white cards, and 3 blue cards is shuffled and dealt into three rows of three cards. In how many ways can we place the cards such that no red cards are in the first row, no white cards are in the second row and no blue cards are in the third row?

SPACE FOR ROUGH WORK

16. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD. Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the integer nearest to BD^4.

17. How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

18. In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, let the area of quadrilateral $MARE$ be $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that b is square-free and a, c are relatively prime. Find $a + b + c$.

19. Triangle NAP has side lengths $NA = 10, AP = 17$, and $NP = 21$. R and S are on $\overline{NP}$ such that $\overline{AR}$ is the altitude to segment $\overline{NP}$ and $\overline{AS}$ is the angle bisector from A to $\overline{NP}$. Let $RS = \frac{m}{n}$ where m, n are relatively prime positve integers. Find $m + n$.

20. Let ABC be a triangle where $AB = 9, BC = 10, CA = 17$. Let Ω be its circumcircle, and let A_1, B_1, C_1 be the diametrically opposite points from A, B, C, respectively, on Ω. Find the sum of digits of the integer nearest to the area of the convex hexagon with the vertices A, B, C, A_1, B_1, C_1.

SPACE FOR ROUGH WORK

21. Define $P = \{\mathrm{S}, \mathrm{T}\}$ and let $\mathcal{P}$ be the set of all proper subsets of P. (A proper subset is a subset that is not the set itself.) How many ordered pairs $(\mathcal{S}, \mathcal{T})$ of proper subsets of $\mathcal{P}$ are there such that

(a) $\mathcal{S}$ is not a proper subset of $\mathcal{T}$ and $\mathcal{T}$ is not a proper subset of $\mathcal{S}$; and

(b) for any sets $S \in \mathcal{S}$ and $T \in \mathcal{T}$, S is not a proper subset of T and T is not a proper subset of S?

22. Bob is coloring lattice points in the coordinate plane. Find the number of ways Bob can color five points in $\{(x, y) \mid 1 \leq x, y \leq 5\}$ blue such that the distance between any two blue points is not an integer.

23. Five people are at a party. Each pair of them are friends, enemies, or frenemies (which is equivalent to being both friends and enemies). It is known that given any three people A, B, C:

- If A and B are friends and B and C are friends, then A and C are friends;
- If A and B are enemies and B and C are enemies, then A and C are friends;
- If A and B are friends and B and C are enemies, then A and C are enemies.

How many possible relationship configurations are there among the five people?

24. How many ways are there to partition the set $\{1, 2, \ldots, 11\}$ into two sets U and V with size 4 and 7 respectively such that the probability that a number chosen from U uniformly at random is greater than a number chosen from V uniformly at random is exactly $\frac{1}{2}$?

25. Triangle $\triangle ABC$ has $AB = 21, BC = 55$, and $CA = 56$. There are two points P in the plane of $\triangle ABC$ for which $\angle BAP = \angle CAP$ and $\angle BPC = 90^\circ$. Find the integer nearest to the distance between them.

SPACE FOR ROUGH WORK

26. A 5 by 5 grid of unit squares is partitioned into 5 pairwise incongruent rectangles with sides lying on the gridlines. Find the sum of square of digits of the maximum possible value of the product of their areas.

27. How many ways can one tile a 2×10 board with 1×1 and 2×2 tiles? Rotations and reflections of the same configuration are considered distinct.

28. Compute the number of integers $n \in \{1, 2, \ldots, 300\}$ such that n is the product of two distinct primes, and is also the length of the longest leg of some nondegenerate right triangle with integer side lengths.

29. For positive reals p and q, define the remainder when p is divided by q as the smallest nonnegative real r such that $\frac{p-r}{q}$ is an integer. For an ordered pair (a, b) of positive integers, let r_1 and r_2 be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs (a, b) such that $a, b \leq 20$ and $r_1 + r_2 = \sqrt{2}$.

30. Square $ARML$ has unit side lengths and center $O.D$ and E are on $\overline{AR}$, with D between A and E. If $\angle DOE = 45°$ and $DE = \frac{3}{7}$, let the largest value of ER be $\frac{a+\sqrt{b}}{c}$ where a, b, c are positive integers such that $a + b + c$ is minimal. Find $a + b + c$.

SPACE FOR ROUGH WORK