MOMC IOQM Mock Donut 1

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources (which can be found in the answer key PDF).
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. The least solution to the equation $8x^3 - 6x = \sqrt{3}$ can be expressed in the form $\cos(k^\circ)$, where $90 < k < 180$. Compute $180 - k$.

2. In rectangle $ABCD$, point E lies on $\overline{AB}$ so that $BE = 8$ and $DE = 20$. Given that $[ABCD] = 288$ and $[AED] = 96$, compute $AD + AE$.

3. Given that $\sqrt{x+y} = 697$, $\sqrt{y+z} = 672$, and $\sqrt{x+z} = 680$, find the integer nearest to

$$\sqrt{\sqrt{x-y} + \sqrt{y-z} + \sqrt{x-z}}$$

4. Mabel has a fair six-sided die with faces labeled $1, 2, 3, 4, m, n$, where m and n are integers with $1 \leq m < n \leq 6$. She rolls this die twice. Let (m, n) be the ordered pair for which the probability that the sum of the two rolls equals 6 is maximized. Find $m^2 + n^2$.

SPACE FOR ROUGH WORK

5. The polynomials $P(x)$ and $Q(x)$ have real coefficients. When $P(x)$ is divided by $(x+1)(x+10)$, the quotient is $Q(x)$ and the remainder is $2x$. When $P(x)$ is divided by $(x+4)(x+7)$, the quotient is $Q(x)$ and the remainder is $38x+18$. Let the remainder when $P(x)$ is divided by $(x-1)$ be N. Find $|N|$.

6. Let x, y, and z be positive integers such that $x+y=20$ and $x+yz=24$. Compute the number of possible ordered triples (x, y, z).

7. Suppose that rectangle $ARML$ with $AR < RM$ has diagonals that intersect at X. Rectangle $AXIS$ is constructed so that X, R, and S are collinear, with R between X and S. Given that $\frac{[AXIS]}{[ARML]} = \frac{23}{24}$, let $\left(\frac{AR}{RM}\right)^2 = \frac{a}{b}$ where a, b are relatively prime positive integers. Find $a+b$.

8. An increasing sequence of 3-digit positive integers satisfies the following properties:

- Each number is a multiple of 2, 3, or 5
- Adjacent numbers differ by only one digit and are relatively prime.

What is the maximum possible length of the sequence?

SPACE FOR ROUGH WORK

9. Let a, b, c be positive real numbers such that $a+b+c = 10$ and $ab+bc+ca = 25$. Let $m = \min\{ab, bc, ca\}$. Let M be the largest possible value of m. Find the integer nearest to $10M$.

10. Let ABC be a triangle with orthocenter H; suppose that $AB = 13, BC = 14, CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Let the area of triangle $G_AG_BG_C$ be $\frac{m}{n}$ where $\gcd(m, n) = 1$. Find $m + n$.

11. Let (a_1, a_3, a_5, a_7) be a permutation of $(1, 4, 9, 16)$, and let (a_2, a_4, a_6, a_8) be a permutation of $(1, 8, 27, 64)$. Determine the last two digits of the greatest possible value of

$$|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + |a_4 - a_5| + |a_5 - a_6| + |a_6 - a_7| + |a_7 - a_8| + |a_8 - a_1| .$$

12. Marcelo and Marta play a game in which a fair coin is flipped repeatedly. The coin either lands heads (H) or tails (T). They play until three consecutive flips appear in the order HTH or HHH. If HTH appears first, then Marcelo wins; otherwise, Marta wins. Compute the number of distinct sequences of coin flips that result in Marcelo winning on the tenth flip.

SPACE FOR ROUGH WORK

13. Let $IOWACTY$ be a regular heptagon. The lines OI and TY intersect at P. Let

$$\frac{[IWY]^2}{[IPY] \cdot [IWCY]}.$$

be written as $\frac{m}{n}$ where $\gcd(m, n) = 1$. Find $m + n$.

14. Eddy and Sarah are at an ARML coaches dinner together, and the menu has 7 different dishes. Let the number of ways that Eddy and Sarah can order dinner so that each of them orders at least one dish, no dish can be ordered twice, and no dish is ordered by both Eddy and Sarah be $100a + b$ where $a, b < 100$ are positive integers. Find $a + b$.

15. Let the number of unordered triples of distinct points in the $4 \times 4 \times 4$ lattice grid $\{0, 1, 2, 3\}^3$ that are collinear in $\mathbb{R}^3$ be N (i.e. there exists a line passing through the three points). Find the largest prime factor of N.

16. Find the maximum possible value of $H \cdot M \cdot M \cdot T$ over all ordered triples (H, M, T) of integers such that $H \cdot M \cdot M \cdot T = H + M + M + T$.

SPACE FOR ROUGH WORK

17. Find the smallest integer $n \geq 5$ for which there exists a set of n distinct pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of positive integers with $1 \leq x_i, y_i \leq 4$ for $i = 1, 2, \ldots, n$, such that for any indices $r, s \in \{1, 2, \ldots, n\}$ (not necessarily distinct), there exists an index $t \in \{1, 2, \ldots, n\}$ such that 4 divides $x_r + x_s - x_t$ and $y_r + y_s - y_t$.

18. In triangle $ABC, AB = 2, AC = 1 + \sqrt{5}$, and $\angle CAB = 54°$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD, determine the measure of $\angle ACM$, in degrees.

19. The arithmetic mean of two positive integers x and y, each less than 100, is 4 more than their geometric mean. Given $x > y$, compute the last two digits of the sum of all possible values for $x + y$. (Note that the geometric mean of x and y is defined to be $\sqrt{xy}$.)

20. $ABCD$ is a cyclic quadrilateral with sides $AB = 10, BC = 8, CD = 25$, and $DA = 12$. A circle ω is tangent to segments DA, AB, and BC. Find the radius of ω.

SPACE FOR ROUGH WORK

21. Triangle ABC has $AB = 13, AC = 14$, and $BC = 15$, and point P is selected randomly and uniformly on $\overline{AC}$. Let $\frac{m}{n}$ be the probability that the distance between the circumcenters of triangles ABP and BCP is at least $5\sqrt{2}$, where m and n are relatively prime positive integers. Find $m+n$.

22. Let a, b, and c be distinct complex numbers such that $a+b+c \neq 0$. Compute the greatest possible number of unordered pairs of different polynomials in the set

$$\left\{ax^2+bx+c, ax^2+cx+b, bx^2+cx+a, bx^2+ax+c, cx^2+ax+b, cx^2+bx+a\right\}$$

that have at least one common root.

23. Compute the greatest integer k such that 2^k divides

$$\sum_{0 \leq i < j \leq 2024} \left[\binom{2024}{i}\binom{2034}{j} - \binom{2024}{j}\binom{2034}{i}\right]^2.$$

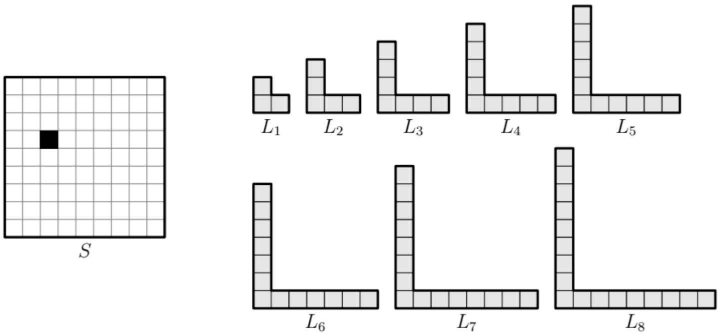
24. Let the number of ordered pairs (a, b) of integers with $1 \leq a \leq b \leq 42$ such that a^b and b^a have the same remainder when divided by 7 be N. Find the largest prime factor of N.

SPACE FOR ROUGH WORK

25. Let S be the set of all positive integers n such that the numbers of divisors of $8n, 9n$, and $10n$ form an arithmetic sequence (in some order) and such that n is not a multiple of 3. Let the second least element of S be M. Find the integer nearest to $\sqrt{M}$.

26. Circle γ has center O and radius 9. Circle ω_A has radius 2 and is internally tangent to γ at point A. Circle ω_B has radius 3 and is internally tangent to γ at point B. The common external tangents of ω_A and ω_B meet at T. Given that $TO = 2 \cdot AB$, let AB be written as $\frac{a\sqrt{b}}{c}$. Find the minimal value of $a+b+c$.

27. In the left figure below, the cell in the fourth row and third column of a 9×9 grid is deleted to obtain a set S of 80 cells. For each $n = 1, 2, 3, \ldots, 8$, an L-shaped tile L_n consisting of $2n+1$ cells is given, as shown in the figure at right.



Let the number of ways to tile S using these eight L-shaped tiles, each exactly once, with rotations of tiles allowed be $100m + n$ where $m, n < 100$ are positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

28. Let the least positive integer satisfying the below two properties be N.

- There exists a positive integer a such that $N = a(2a - 1)$.
- The sum $1 + 2 + \cdots + (N - 1)$ is divisible by k for every integer $1 \leq k \leq 10$.

Find the number of divisors of N.

29. Let S be the set of all 3-digit numbers with all digits in the set $\{1, 2, 3, 4, 5, 6, 7\}$ (so in particular, all three digits are nonzero). For how many elements $\overline{abc}$ of S is it true that at least one of the (not necessarily distinct) "digit cycles"

$$\overline{abc}, \overline{bca}, \overline{cab}$$

is divisible by 7? (Here, $\overline{abc}$ denotes the number whose base 10 digits are a, b, and c in that order.)

30. Let $S = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. A subset P of S is called squarely if it is nonempty and the sum of its elements is a perfect square. A squarely set Q is called super squarely if it is not a proper subset of any squarely set. Find the number of super squarely sets. (A set A is said to be a proper subset of a set B if A is a subset of B and $A \neq B$.)

SPACE FOR ROUGH WORK