MOMC IOQM Mock Donut 2

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources (which can be found in the answer key PDF).
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Equilateral triangles ABF and BCG are constructed outside regular pentagon $ABCDE$. Compute $\angle FEG$.

2. Find the greatest two-digit positive integer x such that for all three-digit (base 10) positive integers $\underline{abc}$, if $\underline{abc}$ is a multiple of x, then the three-digit (base 10) number $\underline{bca}$ is also a multiple of x.

3. Let $ABCD$ be a cyclic quadrilateral, with $AB = 7$, $BC = 11$, $CD = 13$, and $DA = 17$. Let the incircle of ABD hit BD at R and the incircle of CBD hit BD at S. What is RS?

4. Positive integers a, b, and c have the property that a^b, b^c, and c^a end in 4, 2 and 9, respectively. Find the minimum possible value of $a + b + c$.

5. Let $A_1A_2 \ldots A_{19}$ be a regular nonadecagon. Lines A_1A_5 and A_3A_4 meet at X. Find the integer nearest to $\angle A_7XA_5$.

SPACE FOR ROUGH WORK

6. In triangle ABC, points M and N are the midpoints of AB and AC, respectively, and points P and Q trisect BC. Given that A, M, N, P, and Q lie on a circle and $BC = 1$, let the area of triangle ABC be written as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that m, n are relatively prime. Find $m + n$.

7. Let the number of quadruples (a, b, c, d) of positive integers satisfying

$$12a + 21b + 28c + 84d = 2024$$

be M. Find the largest prime factor of M.

8. Let (x, y) be the unique ordered pair of real numbers satisfying the system of equations

$$\frac{x}{\sqrt{x^2 + y^2}} - \frac{1}{x} = 7 \text{ and } \quad \frac{y}{\sqrt{x^2 + y^2}} + \frac{1}{y} = 4.$$

If $|x| = \frac{r}{s}$ and $|y| = \frac{t}{u}$ where r, s, t, u are positive integers such that $\gcd(r, s) = 1$ and $\gcd(t, u) = 1$, find $r + s - t - u$.

SPACE FOR ROUGH WORK

9. Let ABC be a triangle with $\angle BAC = 90°$. Let D, E, and F be the feet of altitude, angle bisector, and median from A to BC, respectively. If $DE = 3$ and $EF = 5$, compute the length of BC.

10. A *lame king* is a chess piece that can move from a cell to any cell that shares at least one vertex with it, except for the cells in the same column as the current cell. A lame king is placed in the top-left cell of a 7×7 grid. Find the maximum number of cells it can visit without visiting the same cell twice (including its starting cell).

11. An ordered pair (a, b) of positive integers is called spicy if $\gcd(a + b, ab + 1) = 1$. Find the number of positive integers $n < 100$ such that both $(99, n)$ and $(101, n)$ are spicy.

12. Let ℓ and m be two non-coplanar lines in space, and let P_1 be a point on ℓ. Let P_2 be the point on m closest to P_1, P_3 be the point on ℓ closest to P_2, P_4 be the point on m closest to P_3, and P_5 be the point on ℓ closest to P_4. Given that $P_1P_2 = 5$, $P_2P_3 = 3$, and $P_3P_4 = 2$, let P_4P_5 be written as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that m, n are relatively prime. Find $m + n$.

SPACE FOR ROUGH WORK

13. Let ABC be a triangle. Let X be the point on side $\overline{AB}$ such that $\angle BXC = 60^\circ$. Let P be the point on segment $\overline{CX}$ such that $BP \perp AC$. Given that $AB = 6, AC = 7$, and $BP = 4$, let $CP = a - \sqrt{b}$ where a, b are positive integers. Find $a + b$.

14. Find the number of ways there are to assemble 2 red unit cubes and 25 white unit cubes into a $3 \times 3 \times 3$ cube such that red is visible on exactly 4 faces of the larger cube. (Rotations and reflections are considered distinct.)

15. Alice, Bob, and Charlie are playing a game with 6 cards numbered 1 through 6. Each player is dealt 2 cards uniformly at random. On each player's turn, they play one of their cards, and the winner is the person who plays the median of the three cards played. Charlie goes last, so Alice and Bob decide to tell their cards to each other, trying to prevent him from winning whenever possible. Let the probability that Charlie wins regardless be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

16. In triangle ABC, a circle ω with center O passes through B and C and intersects segments $\overline{AB}$ and $\overline{AC}$ again at B' and C', respectively. Suppose that the circles with diameters BB' and CC' are externally tangent to each other at T. If $AB = 18, AC = 36$, and $AT = 12$, let $AO = \frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

17. $a \uparrow\uparrow b$ is given by the recurrence

$$a \uparrow\uparrow b = \begin{cases} a & b = 1 \\ a^{a\uparrow\uparrow(b-1)} & b \geq 2 \end{cases}$$

What is the remainder of $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3))$ when divided by 60?

18. Let $ABCD$ be a convex trapezoid such that $\angle DAB = \angle ABC = 90^\circ, DA = 2, AB = 3$, and $BC = 8$. Let ω be a circle passing through A and tangent to segment $\overline{CD}$ at point T. Suppose that the center of ω lies on line BC. Let $CT = a\sqrt{b} - c$ where a, b, c are positive integers such that $a + b + c$ is minimal. Find $a + b + c$.

19. Let $\varphi(n)$ be the number of positive integers less than or equal to n that are coprime to n. Let $\varphi^k(n) = \underbrace{(\varphi \circ \cdots \circ \varphi)}_{k}(n)$ be φ composed with itself k times. Define $\theta(n) = \min\left\{k \in \mathbb{N} \mid \varphi^k(n) = 1\right\}$. For example,

$$\begin{aligned} \varphi^1(13) &= \varphi(13) = 12 \\ \varphi^2(13) &= \varphi(\varphi(13)) = 4 \\ \varphi^3(13) &= \varphi(\varphi(\varphi(13))) = 2 \\ \varphi^4(13) &= \varphi(\varphi(\varphi(\varphi(13)))) = 1 \end{aligned}$$

so $\theta(13) = 4$. Let $f(r) = \theta\left(13^r\right)$. Find the greatest positive integer N such that $f(N) < 100$.

SPACE FOR ROUGH WORK

20. There exists a unique polynomial P in two variables such that for all positive integers m and n,

$$P(m, n) = \sum_{i=1}^{m} \sum_{j=1}^{n} (i + j)^7.$$

Let $|P(3, -3)| = M$. Find the sum of digits of M.

21. The country of HMMTLand has 8 cities. Its government decides to construct several two-way roads between pairs of distinct cities. After they finish construction, it turns out that each city can reach exactly 3 other cities via a single road, and from any pair of distinct cities, either exactly 0 or 2 other cities can be reached from both cities by a single road. Find the sum of digits of the number of ways HMMTLand could have constructed the roads.

22. In each cell of a 4×4 grid, one of the two diagonals is drawn uniformly at random. Let $\frac{p}{q}$ be the probability that the resulting 32 triangular regions can be colored red and blue so that any two regions sharing an edge have different colors where p, q are relatively prime positive integers. Find the sum of digits of $p + q$.

23. Let $P(n) = \left(n - 1^3\right)\left(n - 2^3\right)\ldots\left(n - 40^3\right)$ for positive integers n. Suppose that d is the largest positive integer that divides $P(n)$ for every integer $n > 2023$. If d is a product of m (not necessarily distinct) prime numbers, compute m.

SPACE FOR ROUGH WORK

24. Nine distinct positive integers summing to 74 are put into a 3×3 grid. Simultaneously, the number in each cell is replaced with the sum of the numbers in its adjacent cells. (Two cells are adjacent if they share an edge.) After this, exactly four of the numbers in the grid are 23. Find the greatest positive integer that could have been originally in the center of the grid.

25. Let $a, b, c, d, (a+b+c+18+d), (a+b+c+18-d), (b+c)$, and $(c+d)$ be distinct prime numbers such that $a+b+c = 2010, a, b, c, d \neq 3$, and $d \leq 50$. Find the sum of square of digits of the maximum value of the difference between two of these prime numbers.

26. You are at one vertex of a equilateral triangle with side length 1. All of the edges of the equilateral triangle will reflect the laser beam perfectly (angle of incidence is equal to angle of reflection). Given that the laser beam bounces off exactly 137 edges and returns to the original vertex without touching any other vertices, let M be the maximum possible distance the beam could have traveled, and m be the minimum possible distance the beam could have traveled. Find the integer closest to $\sqrt{M^2 - m^2}$.

27. Let $ABTCD$ be a convex pentagon with area 22 such that $AB = CD$ and the circumcircles of triangles TAB and TCD are internally tangent. Given that $\angle ATD = 90^\circ, \angle BTC = 120^\circ, BT = 4$, and $CT = 5$, compute the integer nearest to the area of triangle TAD.

SPACE FOR ROUGH WORK

28. For each prime p, a polynomial $P(x)$ with rational coefficients is called p-good if and only if there exist three integers a, b, and c such that $0 \leq a < b < c < \frac{p}{3}$ and p divides all the numerators of $P(a)$, $P(b)$, and $P(c)$, when written in simplest form. Find the number of ordered pairs (r, s) of rational numbers such that the polynomial $x^3 + 10x^2 + rx + s$ is p-good for infinitely many primes p.

29. Each lattice point with nonnegative coordinates is labeled with a nonnegative integer in such a way that the point $(0, 0)$ is labeled by 0, and for every $x, y \geq 0$, the set of numbers labeled on the points $(x, y), (x, y+1)$, and $(x+1, y)$ is $\{n, n+1, n+2\}$ for some nonnegative integer n. Let S be the set of all possible labels for the point $(2000, 2024)$. Find the integer nearest to $\sqrt{|S|}$ where $|S|$ represents the number of elements in the set S.

30. Suppose point P is inside quadrilateral $ABCD$ such that

$$\begin{aligned}
\angle PAB &= \angle PDA, \\
\angle PAD &= \angle PDC, \\
\angle PBA &= \angle PCB, \text{ and} \\
\angle PBC &= \angle PCD.
\end{aligned}$$

If $PA = 4, PB = 5$, and $PC = 10$, find the integer nearest to the perimeter of $ABCD$.

SPACE FOR ROUGH WORK