MOMC IOQM Mock Donut 3

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources (which can be found in the answer key PDF).
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Let ℓ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2} = \frac{13}{37}$. Then

$$\frac{(1+\ell)^3}{1+\ell^3} = \frac{m}{n},$$

where m and n are positive coprime integers. Find $m+n$.

2. Construction Mayhem University has been on a mission to expand and improve its campus! The university has recently adopted a new construction schedule where a new project begins every two days. CMU officials claim that each project will take exactly three days to complete, but in reality each project will take exactly one more day than the previous one to complete (so the first project takes 3, the second takes 4, and so on.) Suppose the new schedule starts on Day 1. On which day will there first be at least 10 projects in place at the same time?

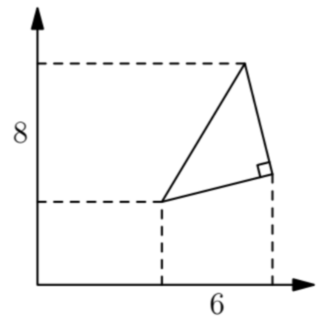
3. Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. The length of $\overline{BC}$ can be written in the form $m\sqrt{n}$ where m and n are positive integers with n as small as possible. What is $m+n$?

4. For a set $S \subseteq \mathbb{N}$, define $f(S) = \{\lceil \sqrt{s} \rceil | s \in S\}$. Suppose there are $100a+b$ sets T such that $|f(T)| = 2$ and $f(f(T)) = \{2\}$, where $a, b < 100$ are positive integers. Find $a+b$.

SPACE FOR ROUGH WORK

5. A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$ respectively. What is the smallest possible value of $a + b$?

6. Right isosceles triangle T is placed in the first quadrant of the coordinate plane. Suppose that the projection of T onto the x-axis has length 6, while the projection of T onto the y-axis has length 8. What is the greatest possible area of the triangle T?



7. For all integers $n \geq 2$, let $f(n)$ denote the largest positive integer m such that $\sqrt[m]{n}$ is an integer. Find the sum of all distinct prime divisors of $f(2) + f(3) + \cdots + f(100)$.

SPACE FOR ROUGH WORK

8. What is the largest positive integer $n < 100$ for which there is a positive integer m satisfying

$$\operatorname{lcm}(m, n) = 3m \times \gcd(m, n)?$$

9. Find the number of ordered triples (a, b, c) of integers that satisfy the equation

$$abc = 2a + 2b + 2c + 4$$

10. Let $f(x) = x^3 - 7x^2 + 16x - 10$. As x ranges over all integers, find the sum of distinct prime values taken on by $f(x)$.

11. Suppose $ABCD$ is a convex quadrilateral satisfying $AB = BC, AC = BD, \angle ABD = 80°$, and $\angle CBD = 20°$. What is $180 - \angle BCD$ in degrees?

12. Suppose integers $a < b < c$ satisfy

$$a + b + c = 95 \quad \text{and} \quad a^2 + b^2 + c^2 = 3083.$$

Find c.

SPACE FOR ROUGH WORK

13. The Pascal Squared Triangle is defined as follows:

- In the n^{th} row, where $n \geq 1$, the first and last elements of the row equal n^2
- Each other element is the sum of the two elements directly above it.

Let S_n denote the sum of the entries in the n^{th} row. Let $\mathcal{N}$ denote the number of integers $1 \leq n \leq 10^6$ such that S_n divisible by 13. Find the sum of digits of $\mathcal{N}$.

14. Find the sum of square of digits of

$$\left| \sum_{1 \leq i < j \leq 50} ij(-1)^{i+j} \right|.$$

15. Point A lies on the circumference of a circle Ω with radius 78. Point B is placed such that AB is tangent to the circle and $AB = 65$, while point C is located on Ω such that $BC = 25$. Compute the length of $\overline{AC}$.

SPACE FOR ROUGH WORK

16. For each integer $n \geq 1$, let S_n be the set of integers $k > n$ such that k divides $30n - 1$. Find the last two digits of the number of of elements in the set

$$\mathcal{S} = \bigcup_{i \geq 1} S_i = S_1 \cup S_2 \cup S_3 \cup \ldots$$

that are less than 2016.

17. For some positive integer n, consider the usual prime factorization

$$n = \prod_{i=1}^{k} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$$

where k is the number of prime factors of n and p_i are the prime factors of n. Define $Q(n), R(n)$ by

$$Q(n) = \prod_{i=1}^{k} p_i^{p_i} \text{ and } R(n) = \prod_{i=1}^{k} e_i^{e_i}.$$

For how many $1 \leq n \leq 70$ does $R(n)$ divide $Q(n)$?

18. Let PQ and PR be tangents to a circle ω with diameter AB so that A, Q, R, B lie on ω in that order. Let H be the projection of P onto AB and let AR and PH intersect at S. If $\angle QPH = 30°$ and $\angle HPR = 20°$, find $\angle ASQ$ in degrees.

SPACE FOR ROUGH WORK

19. Let N be the number of ways to place 4 bishops on a 5×5 chessboard such that no 3 are on the same diagonal. Find the remainder when N is divided by 100. (Note: the length of a diagonal on a 5×5 chessboard can be 2, 3, 4, or 5.)

20. A function $S(m, n)$ satisfies the initial conditions $S(1, n) = n, S(m, 1) = 1$, and the recurrence $S(m, n) = S(m-1, n)S(m, n-1)$ for $m \geq 2, n \geq 2$. Find the largest integer k such that 2^k divides $S(7, 7)$.

21. Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2 + 1} + \frac{1}{r_2^2 + 1} + \cdots + \frac{1}{r_{20}^2 + 1}$$

can be written in the form $\frac{m}{n}$ where m and n are positive coprime integers, find the largest prime factor of $m + n$.

22. Let a, b, c, and d be positive real numbers which satisfy the system of equations

$$\begin{aligned}(a+b)(c+d) &= 143,\\ (a+c)(b+d) &= 150\\ (a+d)(b+c) &= 169.\end{aligned}$$

Let s be the smallest possible value of $a^2 + b^2 + c^2 + d^2$. Find $\lceil \frac{s}{10} \rceil$.

SPACE FOR ROUGH WORK

23. Let $\lfloor x \rfloor$ denote the greatest integer function and $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Let $1 \leq x_1 < \ldots < x_{24}$ be the 24 smallest values of $x \geq 1$ such that $\sqrt{\lfloor x \rfloor \lfloor x^3 \rfloor} + \sqrt{\{x\}\{x^3\}} = x^2$. Find

$$\sum_{k=1}^{12} \frac{1}{x_{2k}^2 - x_{2k-1}^2}.$$

24. In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ while $PD = 821$, what is the integer nearest to $\sqrt{AC}$?

25. Let ABC be a triangle with incenter I and incircle ω. It is given that there exist points X and Y on the circumference of ω such that $\angle BXC = \angle BYC = 90^\circ$. Suppose further that X, I, and Y are collinear. If $AB = 80$ and $AC = 97$, compute the length of BC.

26. Let $\{x\}$ denote the fractional part of x. For example, $\{5.5\} = 0.5$. Find the smallest prime p such that the inequality

$$\sum_{n=1}^{p^2} \left\{ \frac{n^p}{p^2} \right\} > 2016$$

holds.

SPACE FOR ROUGH WORK

27. Compute the number of positive integers $n \leq 50$ such that there exist distinct positive integers a, b satisfying

$$\frac{a}{b} + \frac{b}{a} = n\left(\frac{1}{a} + \frac{1}{b}\right).$$

28. Let ℓ_1 and ℓ_2 be two parallel lines, a distance of 15 apart. Points A and B lie on ℓ_1 while points C and D lie on ℓ_2 such that $\angle BAC = 30°$ and $\angle ABD = 60°$. The minimum value of $AD + BC$ is $a\sqrt{b}$, where a and b are integers and b is square-free. Find $a + b$.

29. Suppose there are M permutations π of $\{1, 2, \ldots, 9\}$ such that there exists an integer N such that $N \equiv \pi(i) \pmod{i}$ for all integers $1 \leq i \leq 9$. Find the sum of square of digits of M.

30. Shen, Ling, and Ru each place four slips of paper with their name on it into a bucket. They then play the following game: slips are removed one at a time, and whoever has all of their slips removed first wins. Shen cheats, however, and adds an extra slip of paper into the bucket, and will win when four of his are drawn. Given that the probability that Shen wins can be expressed as simplified fraction $\frac{m}{n}$, find $|m - n|$.

SPACE FOR ROUGH WORK