MOMC IOQM Mock Donut 4

Instructions:

- All answers should be in the integer range of 00 to 99. Although there is a non-zero chance of an intentional bonus.
- Problems $1-10$ are 2 Markers, $11-20$ are 3 Markers and $21-30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins on the next page. Please proceed to the next page only if you are ready to start the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources, which can be found in the answer key PDF.
- After finishing the mock and checking your answers, **please be sure to fill in this form about the mock.** I really appreciate it!
- Good luck!

1. Find the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Find the integer nearest to the greatest possible difference between the mean and the median scores.

3. Let S be the set of points (x, y) whose coordinates satisfy $x^2 + y^2 \leq 36$ and $(\max\{x, y\})^2 \leq 27$. Find the greatest integer less than the perimeter of S.

4. Given that a, b, and c are positive integers such that $a + bc = 20$ and $a + b = 18$, compute the least possible value of abc.

5. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Find the sum of the digits of the sum of the digits that remain on the chalkboard.

SPACE FOR ROUGH WORK

6. Consider a permutation $(a_1, a_2, a_3, a_4, a_5)$ of $\{1, 2, 3, 4, 5\}$. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is flawless if for all $1 \le i < j < k \le 5$, the sequence (a_i, a_j, a_k) is not an arithmetic progression (in that order). Find the number of flawless 5-tuples.

7. What is the smallest positive integer n which cannot be written in any of the following forms?

- $n = 1 + 2 + \cdots + k$ for a positive integer k.
- $n = p^k$ for a prime number p and integer k.
- $n = p + 1$ for a prime number p.
- $n = pq$ for some distinct prime numbers p and q

8. How many ways can the eight vertices of a three-dimensional cube be colored red and blue such that no two points connected by an edge are both red? Rotations and reflections of a given coloring are considered distinct.

SPACE FOR ROUGH WORK

9. Seven little children sit in a circle. The teacher distributes pieces of candy to the children in such a way that the following conditions hold.

- Every little child gets at least one piece of candy.
- No two little children have the same number of pieces of candy.
- The numbers of candy pieces given to any two adjacent little children have a common factor other than 1.
- There is no prime dividing every little child's number of candy pieces.

What is the smallest number of pieces of candy that the teacher must have ready for the little children?

10. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved 360° about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Let $k = a\sqrt{b}$ where a, b are positive integers such that b is square-free. Find the largest prime factor of a.

11. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, let the probability that $\tan\theta$ is rational be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.

SPACE FOR ROUGH WORK

12. Find the product of all positive values of x that satisfy

$$\lfloor x+1 \rfloor^{2x} - 19\lfloor x+1 \rfloor^{x} + 48 = 0.$$

13. Triangle ABC is inscribed in circle ω. The line containing the median from A meets ω again at M, the line containing the angle bisector of $\angle B$ meets ω again at R, and the line containing the altitude from C meets ω again at L. Given that quadrilateral $ARML$ is a rectangle, find the degree measure of $\angle BAC$.

14. Find the number of divisors of the least four-digit integer N for which N and $N + 2018$ contain a total of 8 distinct digits.

15. The increasing infinite arithmetic sequence of integers $x_1, x_2, x_3, \ldots$ contains the terms $17!$ and $18!$. Find the greatest integer X for which $X!$ must also appear in the sequence.

16. For any positive integer x, define $\text{Accident}(x)$ to be the set of ordered pairs (s, t) with $s \in \{0, 2, 4, 5, 7, 9, 11\}$ and $t \in \{1, 3, 6, 8, 10\}$ such that $x + s - t$ is divisible by 12. For any nonnegative integer i, let a_i denote the number of $x \in \{0, 1, \ldots, 11\}$ for which $|\text{Accident}(x)| = i$. Find

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2.$$

SPACE FOR ROUGH WORK

17. Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 24, \quad \text{and} \quad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x$$

Find $|x|$.

18. Let ABC be a triangle with $AB = 5, AC = 4, BC = 6$. The angle bisector of C intersects side AB at X. Points M and N are drawn on sides BC and AC, respectively, such that $\overline{XM} \| \overline{AC}$ and $\overline{XN} \| \overline{BC}$. Let the length MN be $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that $a+b+c$ is minimal and b is square-free. Find $a+b+c$.

19. Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the smallest integer greater than the maximum possible value of $x^2 + xy + 2y^2$.

20. Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω. Suppose that BD and AC intersect at a point E. The tangent to ω at B meets line AC at a point F, so that C lies between E and F. Given that $AE = 6, EC = 4, BE = 2$, and $BF = 12$, let $DA = p\sqrt{q}$ where p, q are positive integers such that q is square-free. Find $p+q$.

SPACE FOR ROUGH WORK

21. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Let the least upper bound of the set of distances between points in S be $a\sqrt{b}$ where a, b are positive integers such that b is square-free. Find $a + b$.

22. Find the sum of digits of the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

23. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1, BC = 4, CD = 2$, and $AD = 7$, find the integer nearest to the sum of the areas of ω_1 and ω_2.

SPACE FOR ROUGH WORK

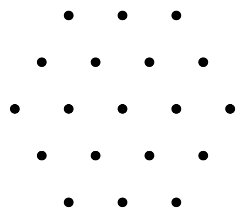
24. Find the largest prime factor of the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

25. Let $\mathcal{S}_1 = \{1, 2, 3, 4\}, \mathcal{S}_2 = \{3, 4, 5, 6\}$, and $\mathcal{S}_3 = \{6, 7, 8\}$. Let $100a + b$ be the number of sets $\mathcal{H}$ that satisfy both of the following properties:

- $\mathcal{H} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$;
- each of $\mathcal{H} \cap \mathcal{S}_1, \mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is non-empty.

where $a, b < 100$ are positive integers. Find $a + b$.

26. How many lines pass through exactly two points in the following hexagonal grid?



SPACE FOR ROUGH WORK

27. Suppose that $(a_1, \ldots, a_{20})$ and $(b_1, \ldots, b_{20})$ are two sequences of integers such that the sequence $(a_1, \ldots, a_{20}, b_1, \ldots, b_{20})$ contains each of the numbers $1, \ldots, 40$ exactly once. Suppose the maximum possible value of the sum

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j)?$$

is $100a + b$ where $a, b < 100$ are positive integers. Find $a + b$.

28. Let $A = \{a_1, a_2, \ldots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Let the smallest possible value of the sum of the elements in A be $\mathcal{S}$. Find the integer nearest to $\sqrt{\mathcal{S}}$.

29. Let $a_1, a_2, \ldots$ be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is sum of digits of the maximal number of distinct terms in the sequence $b_1, b_2, \ldots$?

30. Let n be a positive integer. A sequence $(a_0, \ldots, a_n)$ of integers is acceptable if it satisfies the following conditions:

(a) $0 = |a_0| < |a_1| < \cdots < |a_{n-1}| < |a_n|$.

(b) The sets $\{|a_1 - a_0|, |a_2 - a_1|, \ldots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{1, 3, 9, \ldots, 3^{n-1}\}$ are equal.

Let the number of acceptable sequences of integers be $f(n)$. Find the greatest positive integer n such that $f(n) < 2024$.

SPACE FOR ROUGH WORK