MOMC IOQM Mock Espresso 1

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources (which can be found in the answer key PDF).
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Find the number of ordered triples of positive integers (a, b, c) such that $\sqrt{a^b + c!} = 28$.

2. Given that C, A, T, F, I, S, and H are digits, not necessarily distinct, and that

$$3 \cdot \underline{CAT} \cdot \underline{FISH} = \underline{CATFISH},$$

let the sum of square of digits of the greatest possible value of $\underline{CATFISH}$ be N. Find the product of digits of N.

3. Ari repeatedly rolls a standard, fair, six-sided die. Let $R(n)$ be the n^{th} number rolled, and let $Q(n) = R(1) \cdot R(2) \cdot \ldots \cdot R(n)$. Let the probability that there exists an n such that $Q(n) = 100$ and for all $m < n, Q(m)$ is not a perfect square be $\frac{p}{q}$ where p, q are relatively prime positive integers. Find the largest prime factor of $p + q$.

4. Trapezoid $ARML$ has $\overline{AR} \| \overline{ML}$. Given that $AR = 4, RM = \sqrt{26}, ML = 12$, and $LA = \sqrt{42}$, find the integer nearest to AM^2.

SPACE FOR ROUGH WORK

5. In $\triangle ABC, \angle A = 90^\circ, AC = 1$, and $AB = 5$. Point D lies on ray $\overrightarrow{AC}$ such that $\angle DBC = 2\angle CBA$. Let $AD = \frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

6. In triangle $ABC, \angle C = 90^\circ$ and $BC = 17$. Point E lies on side $\overline{BC}$ such that $\angle CAE = \angle EAB$. The circumcircle of triangle ABE passes through a point F on side $\overline{AC}$. Given that $CF = 3$, find the integer nearest to AB.

7. Find the difference between the largest and the smallest 2-digit prime number p such that there exists a prime number q for which $100q + p$ is a perfect square.

8. Find the number of ways to color 3 cells in a 3×3 grid so that no two colored cells share an edge.

9. Sets A, B, and C satisfy $|A| = 92, |B| = 35, |C| = 63, |A \cap B| = 16, |A \cap C| = 51, |B \cap C| = 19$. Compute the number of possible values of $|A \cap B \cap C|$.

SPACE FOR ROUGH WORK

10. For any positive integer n, let $\tau(n)$ denote the number of positive divisors of n. If n is a positive integer such that $\frac{\tau(n^2)}{\tau(n)} = 3$, compute $\frac{\tau(n^7)}{\tau(n)}$.

11. Let $\{a_n\}$ be a sequence with $a_0 = 1$, and for all $n > 0, a_n = \frac{1}{2}\sum_{i=0}^{n-1} a_i$. Find the greatest value of n for which $a_n < 2024$.

12. Chris the frog begins on a number line at 0. Chris takes jumps of lengths $1, 2, 3, \ldots, 2024$, in that order. If Chris's current location is an even integer, he jumps in the positive direction; otherwise, he jumps in the negative direction. Let $P(n)$ denote Chris's location after the n^{th} jump. Let $S = |\sum_{j=1}^{2024} P(j)|$. Find the largest prime factor of S.

13. Find the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common.

SPACE FOR ROUGH WORK

14. Cube $ARMLKHJC$, with opposite faces $ARML$ and $HJCK$, is inscribed in a cone, such that A is the vertex of the cone, edges $\overline{AR}, \overline{AL}, \overline{AH}$ lie on the surface of the cone, and vertex C, diagonally opposite A, is on the base of the cone. Given that $AR = 6$, find the integer nearest to the radius of the cone.

15. Let $(a_1, a_2, \ldots, a_8)$ be a permutation of $(1, 2, \ldots, 8)$. Find the maximum possible number of elements of the set

$$\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_8\}$$

that can be perfect squares.

16. Given a positive integer k, let $\|k\|$ denote the absolute difference between k and the nearest perfect square. For example, $\|13\| = 3$ since the nearest perfect square to 13 is 16. Let the smallest positive integer n such that

$$\frac{\|1\| + \|2\| + \cdots + \|n\|}{n} = 100$$

be $\mathcal{N}$. Find the sum of digits of $\mathcal{N}$.

SPACE FOR ROUGH WORK

17. Find the number of nonempty subsets $S \subseteq \{-10, -9, -8, \ldots, 8, 9, 10\}$ that satisfy $|S| + \min(S) \cdot \max(S) = 0$.

18. Michel starts with the string $HMMT$. An operation consists of either replacing an occurrence of H with HM, replacing an occurrence of MM with MOM, or replacing an occurrence of T with MT. For example, the two strings that can be reached after one operation are $HMMMT$ and $HMOMT$. Find the number of distinct strings Michel can obtain after exactly 9 operations.

19. A positive integer n is loose it has six positive divisors and satisfies the property that any two positive divisors $a < b$ of n satisfy $b \geq 2a$. Find the number of loose positive integers less than 100.

20. Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the distinct real solutions to the equation

$$\left(x^2 + y^2\right)^6 = \left(x^2 - y^2\right)^4 = \left(2x^3 - 6xy^2\right)^3.$$

Then $\sum_{i=1}^{k} (x_i + y_i)$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

SPACE FOR ROUGH WORK

21. On July 17, 2017, the nation of Armlandia will turn n^2 years old and the nation of Nysmlistan will turn n years old. The next four anniversaries for which Nysmlistan's age divides Armlandia's age will occur, in order, on July 17 in the years 2027, 2032, 2038, and M. Find the sum of square of digits of M.

22. Rhombus $ARML$ has its vertices on the graph of $y = \lfloor x \rfloor - \{x\}$. Given that $[ARML] = 8$, let the least upper bound for $\tan A$ be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find the sum of digits of $m + n$.

23. Find the sum of digits of the least positive N such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

24. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5) be the vertices of a regular pentagon centered at $(0, 0)$. Find the sum of all positive integers k such that the equality

$$\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k$$

must hold for all possible choices of the pentagon.

SPACE FOR ROUGH WORK

25. The numbers $1, 2, \ldots, 10$ are randomly arranged in a circle. Let p be the probability that for every positive integer $k < 10$, there exists an integer $k' > k$ such that there is at most one number between k and k' in the circle. If p can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the sum of square of digits of $a + b$.

26. Find the number of permutations π of the set $\{1, 2, \ldots, 10\}$ so that for all (not necessarily distinct) $m, n \in \{1, 2, \ldots, 10\}$ where $m + n$ is prime, $\pi(m) + \pi(n)$ is prime.

27. Let $a \neq b$ be positive real numbers and m, n be positive integers. An $m + n$-gon P has the property that m sides have length a and n sides have length b. Further suppose that P can be inscribed in a circle of radius $a + b$. Let the number of ordered pairs (m, n), with $m, n \leq 100$, for which such a polygon P exists for some distinct values of a and b be $100p + q$ where $p, q < 100$ are positive integers. Find $p + q$.

SPACE FOR ROUGH WORK

28. Let π be a uniformly random permutation of the set $\{1, 2, \ldots, 100\}$. The probability that $\pi^{20}(20) = 20$ and $\pi^{21}(21) = 21$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the sum of digits of $a + b$. (Here, π^k means π iterated k times.)

29. In the Cartesian plane, let $A = (0, 0), B = (200, 100)$, and $C = (30, 330)$. Find the sum of square of digits of the number of ordered pairs (x, y) of integers so that $\left(x + \frac{1}{2}, y + \frac{1}{2}\right)$ is in the interior of triangle ABC.

30. Let $S = \{(x, y) \in \mathbb{Z}^2 \mid 0 \le x \le 11, 0 \le y \le 9\}$. Find the sum of digits of the number of sequences $(s_0, s_1, \ldots, s_n)$ of elements in S (for any positive integer $n \ge 2$) that satisfy the following conditions:

- $s_0 = (0, 0)$ and $s_1 = (1, 0)$
- $s_0, s_1, \ldots, s_n$ are distinct,
- for all integers $2 \le i \le n, s_i$ is obtained by rotating s_{i-2} about s_{i-1} by either $90°$ or $180°$ in the clockwise direction.

SPACE FOR ROUGH WORK