MOMC IOQM Mock Espresso 2

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources (which can be found in the answer key PDF).
- After finishing the mock and checking your answers, **be sure to fill in this form about the mock.** I will really appreciate it!
- Good luck!

1. Cindy has a collection of identical rectangular prisms. She stacks them, end to end, to form 1 longer rectangular prism. Suppose that joining 11 of them will form a rectangular prism with 3 times the surface area of an individual rectangular prism. How many will she need to join end to end to form a rectangular prism with 9 times the surface area?

2. Let $1 = a_1 < a_2 < a_3 < \ldots < a_k = n$ be the positive divisors of n in increasing order. If $n = a_3^3 - a_2^3$, what is n?

3. A point (x_0, y_0) with integer coordinates is a primitive point of a circle if for some pair of integers (a, b), the line $ax + by = 1$ intersects the circle at (x_0, y_0). How many primitive points are there of the circle centered at $(2, -3)$ with radius 5?

4. Find the sum of all possible values of $|m+n|$ where m, n are integers that satisfy $m^2+2m-35 = 2^n$.

SPACE FOR ROUGH WORK

5. Elizabeth is at a candy store buying jelly beans. Elizabeth begins with 0 jellybeans. With each scoop, she can increase her jellybean count to the next largest multiple of 30, 70 or 110. (For example, her next scoop after 70 can increase her jellybean count to 90, 110, or 140). Let the smallest number of jellybeans Elizabeth can collect in more than 100 different ways be S. Find the integer nearest to $S/10$.

6. One of the six digits in the expression $435 \cdot 605$ can be changed so that the product is a perfect square N^2. Find the remainder when N is divided by 100.

7. Triangle $\triangle ABC$ has $AB = 3, BC = 4$, and $AC = 5$. Let M and N be the midpoints of AC and BC, respectively. If line AN intersects the circumcircle of triangle $\triangle BMC$ at points X and Y, then $XY^2 = \frac{m}{n}$ for some relatively prime positive integers m, n. Find the sum of digits of $m + n$.

SPACE FOR ROUGH WORK

8. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD. Let R_1, R_2, R_3, R_4 be the circumradii of triangles APB, BPC, CPD, DPA respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4.

9. Start by writing the integers $1, 2, 4, 6$ on the blackboard. At each step, write the smallest positive integer n that satisfies both of the following properties on the board.

- n is larger than any integer on the board currently.
- n cannot be written as the sum of 2 distinct integers on the board.

Find the largest integer less than 100 that you write on the board.

10. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC. Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the integer closest to the maximum possible area of the triangle?

11. Positive integer n can be written in the form $a^2 - b^2$ for at least 12 pairs of positive integers (a, b). Find the number of divisors of the smallest possible value of n.

SPACE FOR ROUGH WORK

12. Let P and A denote the perimeter and area respectively of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$.

13. Let $ABCD$ be a square. Point E is chosen inside the square such that $AE = 6$. Point F is chosen outside the square such that $BE = BF = 2\sqrt{5}, \angle ABF = \angle CBE$, and $AEBF$ is cyclic. Find the area of $ABCD$.

14. Let $a_k = \pm 1$ for all integers $1 \leq k \leq 2018$. The sum

$$\sum_{1 \leq i < j \leq 2018} a_i a_j$$

can take on both positive and negative values. Find the smallest positive value of the sum.

15. Let A, B, C, D be points, in order, on a straight line such that $AB = BC = CD$. Let E be a point closer to B than D such that $BE = EC = CD$ and let F be the midpoint of DE. Let AF intersect EC at G and let BF intersect EC at H. If $[BHC] + [GHF] = 1$, then $AD^2 = \frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. Find the sum of square of digits of $a + b + c$.

SPACE FOR ROUGH WORK

16. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Then he finds the sum S of all the integers he wrote down. In other words, he finds

$$S = \sum_{\substack{abc=2310 \\ a,b,c\in\mathbb{N}}} (a + b + c)$$

where $\mathbb{N}$ denotes the positive integers. Determine the sum of all prime factors of S with multiplicity. (For example, such a sum for 12 would be $2 + 2 + 3 = 7$.)

17. Let $ABCD$ be a quadrilateral with side lengths $AB = 2, BC = 3, CD = 5$, and $DA = 4$. Let the maximum possible radius of a circle inscribed in quadrilateral $ABCD$ be $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that $a + b + c$ is minimal. Find $a + b + c$.

18. Sean is a biologist, and is looking at a string of length 66 composed of the letters A, T, C, G. A substring of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has 10 substrings: $A, G, T, C, AG, GT, TC, AGT, GTC, AGTC$. Let the maximum number of distinct substrings of the string Sean is looking at be $100m + n$ where $m, n < 100$ are positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

19. Let ω and Γ by circles such that ω is internally tangent to Γ at a point P. Let AB be a chord of Γ tangent to ω at a point Q. Let $R \neq P$ be the second intersection of line PQ with Γ. If the radius of Γ is 17, the radius of ω is 7, and $\frac{AQ}{BQ} = 3$, find the integer nearest to the circumradius of triangle AQR.

20. Find the sum of digits of the number of 5-digit numbers n that exist such that each n is divisible by 9 and none of the digits of n are divisible by 9?

21. Let (a, b, c, d, e) be an integer solution to the system of equations

$$\begin{aligned} a + d &= 12 \\ b + ad + e &= 57 \\ c + bd + ae &= 134 \\ cd + be &= 156 \\ ce &= 72 \end{aligned}$$

Find the maximal value of $b + d$.

22. Define a permutation of the set $\{1, 2, 3, \ldots, n\}$ to be sortable if upon cancelling an appropriate term of such permutation, the remaining $n - 1$ terms are in increasing order. If $f(n)$ is the number of sortable permutations of $\{1, 2, 3, \ldots, n\}$, find the remainder when

$$\sum_{i=1}^{2024} (-1)^i \cdot f(i)$$

is divided by 100. Note that the empty set is considered sortable.

SPACE FOR ROUGH WORK

23. In circle Ω, let $\overline{AB} = 65$ be the diameter and let points C and D lie on the same side of arc AB such that $CD = 16$, with C closer to B and D closer to A. Moreover, let AD, BC, AC, and BD all have integer lengths. Two other circles, circles ω_1 and ω_2, have $\overline{AC}$ and $\overline{BD}$ as their diameters, respectively. Let circle ω_1 intersect AB at a point $E \neq A$ and let circle ω_2 intersect AB at a point $F \neq B$. Then $EF = \frac{m}{n}$, for relatively prime integers m and n. Find the sum of digits of $m+n$.

24. In a $25 \times n$ grid, each square is colored with a color chosen among 8 different colors. Let n be as minimal as possible such that, independently from the coloration used, it is always possible to select 4 coloumns and 4 rows such that the 16 squares of the intersections are all of the same color. Find the sum of digits of n.

25. Kelvin the Frog and 10 of his relatives are at a party. Every pair of frogs is either friendly or unfriendly. When 3 pairwise friendly frogs meet up, they will gossip about one another and end up in a fight (but stay friendly anyway). When 3 pairwise unfriendly frogs meet up, they will also end up in a fight. In all other cases, common ground is found and there is no fight. If all $\binom{11}{3}$ triples of frogs meet up exactly once, what is the minimum possible number of fights?

26. The equation

$$(x-1)(x-2)(x-4)(x-5)(x-7)(x-8) = (x-3)(x-6)(x-9)$$

has distinct roots $r_1, r_2, \ldots, r_6$. Find the integer nearest to the square root of

$$\sum_{i=1}^{6} (r_i - 1)(r_i - 2)(r_i - 4)$$

SPACE FOR ROUGH WORK

27. Let $\mathcal{P}$ be the set of all polynomials $p(x) = x^4 + 2x^2 + mx + n$, where m and n range over the positive reals. There exists a unique $p(x) \in \mathcal{P}$ such that $p(x)$ has a real root, m is minimized, and $p(1) = 99$. Find n.

28. Mark has six boxes lined up in a straight line. Inside each of the first three boxes are a red ball, a blue ball, and a green ball. He randomly selects a ball from each of the three boxes and puts them into a fourth box. Then, he randomly selects a ball from each of the four boxes and puts them into a fifth box. Next, he randomly selects a ball from each of the five boxes and puts them into a sixth box, arriving at three boxes with 1, 3, and 5 balls, respectively. The probability that the box with 3 balls has each type of color is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

29. Edward has a 3×3 tic-tac-toe board and wishes to color the squares using 3 colors. Suppose that the number of ways in which he can color the board such that there is at least one row whose squares have the same color and at least one column whose squares have the same color be $100a + b$ where $a, b < 100$ are positive integers. Find $|a - b|$. Note that a coloring does not have to contain all three colors and Edward cannot rotate or reflect his board.

30. Let ABC be an acute triangle with $BC = 48$. Let M be the midpoint of BC, and let D and E be the feet of the altitudes drawn from B and C to AC and AB respectively. Let P be the intersection between the line through A parallel to BC and line DE. If $AP = 10$, compute the length of PM.

SPACE FOR ROUGH WORK