MOMC Regional Mathematical Olympiad Mock Orange 1

Time: 3 Hours **October 22, 2023**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
- **Problems collected by Agamjeet Singh**

1. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3, \ldots.$$

2. Given positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1^2 + 2a_2^3 + \cdots + na_n^{n+1} < 1$. Prove that $2a_1 + 3a_2^2 + \cdots + (n+1)a_n^n < 3$.

3. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle BCD = \theta$ for some angle $\theta < 90°$. Point X lies inside the quadrilateral such that $\angle XAD = \angle XDA = 90° - \theta$. Prove that $BX = XC$.

4. Let $-1 < x_1 < x_2, \cdots < x_n < 1$ and $x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n$. Prove that if $y_1 < y_2 < \cdots < y_n$, then

$$x_1^{13} y_1 + \cdots + x_n^{13} y_n < x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

5. Let ABC be a triangle. Point D lies on segment BC such that $\angle BAD = \angle DAC$. Point X lies on the opposite side of line BC as A and satisfies $XB = XD$ and $\angle BXD = \angle ACB$. Analogously, point Y lies on the opposite side of line BC as A and satisfies $YC = YD$ and $\angle CYD = \angle ABC$. Prove that lines XY and AD are perpendicular.

6. Let $\{a_1, a_2, \cdots, a_n\}$ be a permutation of $\{1, 2, 3 \cdots, n\}$. Prove that the sum

$$\sum \frac{1}{(a_1)(a_1 + a_2)(a_1 + a_2 + a_3) \ldots (a_1 + a_2 + \cdots + a_n)}$$

taken over all permutations equals $\frac{1}{n!}$.

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