MOMC Regional Mathematical Olympiad Mock
Orange 3

Time: 3 Hours **October 26, 2023**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
- **Problems collected by Agamjeet Singh**

1. Let ABC be a triangle of which the side lengths are positive integers which are pairwise coprime. The tangent in A to the circumcircle intersects line BC in D. Prove that BD is not an integer.

2. Prove that there doesn't exist any positive integer n such that $2n^2+1, 3n^2+1$ and $6n^2+1$ are perfect squares.

3. Find all monic polynomials $P(x) = x^{2023} + a_{2022}x^{2022} + \ldots + a_1x + a_0$ with real coefficients such that $a_{2022} = 0$, $P(1) = 1$ and all roots of P are real and less than 1.

4. Let $0 < a, b, c < 1$ with $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3\sqrt{3}}{2}.$$

Determine when equality holds.

5. Two circles Γ_1 and Γ_2 intersect at points A and Z (with $A \neq Z$). Let B be the centre of Γ_1 and let C be the centre of Γ_2. The exterior angle bisector of $\angle BAC$ intersects Γ_1 again at X and Γ_2 again at Y. Prove that the interior angle bisector of $\angle BZC$ passes through the circumcenter of $\triangle XYZ$.

6. Find all positive integers m and n that satisfy the equality:

$$n^5 + n^4 = 7^m - 1$$

———0———